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Inequalities on generalized trigonometric and hyperbolic functions

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ABSTRACT

In this paper, we prove some inequalities involving the generalized trigonometric and hyperbolic functions.

1. Introduction

One way to define the trigonometric functions is via integration as follows:

$$\arcsin(x) = \int_0^x \frac{1}{(1-t^2)^{1/2}} dt, \quad 0 \le x \le 1,$$

and

$$\frac{\pi}{2} = \arcsin(1) = \int_0^1 \frac{1}{(1-t^2)^{1/2}} dt, \quad 0 \le x \le 1.$$

We define sin on $[0, \frac{\pi}{2}]$ as the inverse of arcsin. For $1 < p < \infty$, we generalize the inverse sine function as follows:

$$\arcsin_p(x) \equiv \int_0^x \frac{1}{(1-t^p)^{1/p}} dt, \quad 0 \le x \le 1,$$

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and

$$\frac{\pi_p}{2} = \operatorname{arcsin}_p(1) \equiv \int_0^1 \frac{1}{(1-t^p)^{1/p}}dt, \quad 0 \le x \le 1.$$

The inverse of arcsin_p on $[0, \frac{\pi}{2}]$ is called generalized sine function and denoted by $\sin_p$.

The generalized cosine function $\cos_p$ is defined as

$$\cos_p(x) \equiv \frac{d}{dx}\sin_p(x).$$

It is clear from the definition that

$$\cos_p(x) = \left(1 - \sin_p(x)^p\right)^{1/p}, \quad x \in [0, \pi_p/2],$$

and

$$\operatorname{arccos}_p(x) = \operatorname{arcsin}_p\left(\left(1 - x^p\right)^{1/p}\right).$$

The generalized tangent function is defined as in the classical case:

$$\tan_p(x) \equiv \frac{\sin_p(x)}{\cos_p(x)}, \quad x \in \mathbb{R}\setminus\left\{k\pi_p + \frac{\pi_p}{2} : k \in \mathbb{Z}\right\}.$$

Similarly, the generalized inverse hyperbolic sine function $\operatorname{arcsinh}_p$ is defined as

$$\operatorname{arcsinh}_p(x) \equiv \begin{cases} \int_0^x \frac{1}{(1+t^p)^{1/p}}dt, & x \in [0, \infty), \\ -\operatorname{arcsinh}_p(-x), & x \in (-\infty, 0). \end{cases}$$

The inverse of $\operatorname{arcsinh}_p$ is called the generalized hyperbolic sine function and denoted by $\sinh_p$. The generalized hyperbolic cosine function $\cosh_p(x)$ is defined as

$$\cosh_p(x) \equiv \frac{d}{dx}\sinh_p(x).$$

It is clear from the definition that

$$\cosh_p(x) = \left(1 + \sinh_p(x)^p\right)^{1/p}, \quad x \in \mathbb{R},$$

and

$$\operatorname{arccosh}_p(x) = \operatorname{arcsinh}_p\left(\left(x^p - 1\right)^{1/p}\right).$$

The generalized hyperbolic tangent function is defined as

$$\tanh_p(x) \equiv \frac{\sinh_p(x)}{\cosh_p(x)}.$$

It is easy to see that

$$\frac{d}{dx}\cos_p(x) = -\cos_p(x)^{2-p}\sin_p(x)^{p-1}, \quad x \in [0, \pi_p/2],$$