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Fractal perturbation preserving fundamental shapes: Bounds on the scale factors

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ABSTRACT

Fractal interpolation function defined through suitable iterated function system provides a method to perturb a function $f \in \mathcal{C}(I)$ so as to yield a class of functions $f^\alpha \in \mathcal{C}(I)$, where α is a free parameter, called scale vector. For suitable values of scale vector α, the fractal functions f^α simultaneously interpolate and approximate f. Further, the iterated function system can be selected suitably so that the corresponding fractal function f^α shares the quality of smoothness or non-smoothness of f. The objective of the present paper is to choose elements of the iterated function system appropriately in order that f^α preserves fundamental shape properties, namely positivity, monotonicity, and convexity in addition to the regularity of f in the given interval. In particular, the scale factors (elements of the scale vector) must be restricted to satisfy two inequalities that provide numerical lower and upper bounds for the multipliers. As a consequence of this process, fractal versions of some elementary theorems in shape preserving interpolation/approximation are obtained. For instance, positive approximation (that is to say, using a positive function) is extended to the fractal case if the factors verify certain inequalities.

1. Introduction

One of the central themes in numerical analysis/approximation theory is to represent an arbitrary function or a data set in terms of functions which are easier to describe and convenient to use. When we have to deal with irregular forms, for instance, real world signals such as financial series, time series, climate data, and bioelectric recordings, traditional methods may not provide an approximant with a desired precision. Fractal functions form the basis of a constructive approximation theory for non-differentiable functions.

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The idea of interpolation and approximation using fractal methodology first appeared in the work of Barnsley [1,2]. Barnsley introduced fractal functions as continuous functions interpolating a given set of data points. Since its inception, fractal interpolation function has been developed both in theory and applications by many authors, see for example [3,4,6,9,11,12] and references therein.

The methods in fractal approximation theory are based on iterated function system (IFS), which are chosen suitably for different target functions. Given a continuous function f defined on a real compact interval, Barnsley and Navascués have considered suitable IFS to construct continuous functions f^α that simultaneously interpolate and approximate f. The graph of f^α is a union of transformed copies of itself, and, in general, f^α may have noninteger Hausdorff and Minkowski dimensions. Due to these fractal characteristics, f^α may be treated as the fractal perturbation of f. In this way, every continuous function is generalized with a family of fractal functions. The degrees of freedom offered by this procedure may be useful when some problems combined with approximation and optimization have to be approached.

Navascués and group have studied various properties of the fractal perturbation f^α of f and proposed the fractal operator $\mathcal{F}^\alpha : \mathcal{C}(I) \to \mathcal{C}(I)$, where $\mathcal{C}(I)$ denotes the space of continuous functions on a real compact interval I, that maps $f \mapsto f^\alpha$ (see [13–17]). In this fractal perturbation process, these studies concern mainly on two properties, namely smoothness and approximation order among the various desirable properties of a good approximant. To be precise, given a function $f \in \mathcal{C}^p(I)$, it is known how to select the elements of the IFS so that the corresponding fractal function $f^\alpha \in \mathcal{C}^p(I)$. Similarly, for a given original function Φ with its traditional approximant f, we can perturb f using fractal method so as to yield f^α that has the same approximation order as that of f. However, in many problems arising in engineering and science, one requires approximation methods to reproduce physical reality as close as possible. Schematically, given a function or data set with a shape S one desires to represent it by a function that approximates it well, and, in addition, has the same shape S. This kind of approximation is called *shape preserving approximation* and arises quite naturally in fields such as computer aided geometric design, robotics, data visualization, chemical and physical sciences, and reverse engineering. Recently, our group has developed the shape preserving aspects of the cubic Hermite fractal interpolation function (FIF), the rational quadratic FIF and the rational cubic FIF in constructive manner (see [7,8,10]).

The aforementioned considerations naturally lead to the question: can we find fractal perturbation f^α of f that retains properties of the germ f? The current article seeks to develop suitable methods to choose elements of the IFS appropriately so that the corresponding fractal functions f^α retain the order of continuity and the fundamental shape properties, namely positivity, monotonicity, and convexity of f. The selection of the parameters involves the boundedness of the scale factors (components of the scale vector α) of the transformation, by means of two appropriate inequalities. Interpolating a given data set within a prescribed frame is a basic requirement in image compression. The parameter identification problem given in this article can be adapted to interpolate a given function f at specified knot points with the help of a fractal function f^α whose graph is contained in a prescribed axis-aligned rectangle. The method used here is more general than that in the parameter identification problems discussed by Dalla et al. in [5,11], wherein the analysis depends heavily on the affinity of the maps in the IFS.

For the sake of simplicity, we have presented shape preserving aspects of the fractal perturbation with the assumption that the function $f \in \mathcal{C}(I)$ being perturbed possesses uniform shape property on the entire interval (see Theorems 3.1, 4.2, 5.1). However, $f \in \mathcal{C}(I)$ may not have a uniform shape on the entire interval I and we may require the fractal function f^α to preserve the shape of f. For instance, suppose that a function $f \in \mathcal{C}(I)$ switches back and forth between nonnegativity and nonpositivity (say a finite number of times). To obtain a fractal function $f^\alpha \in \mathcal{C}(I)$ that is copositive with f (i.e., $f(x)f^\alpha(x) \geq 0$ for all $x \in I$), we shall proceed as follows. Subdivide I into subintervals I_i, $i = 1, 2, \ldots, r$ such that in a typical subinterval I_s the function f is either nonnegative or nonpositive throughout. In each subinterval I_i, we choose elements of the IFS so as to meet the specifications in Theorem 3.1. Consequently, in each interval I_i we can produce a fractal function f^{α^i} which honours the nonnegativity/nonpositivity of f. Letting α to be