J. Math. Anal. Appl. 419 (2014) 904–937

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

A pre-order principle and set-valued Ekeland variational principle ☆

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ARTICLE INFO

Article history:
Received 11 April 2013
Available online 15 May 2014
Submitted by A. Dontchev

Keywords:
Pre-order principle
Ekeland variational principle
Set-valued map
Perturbation
Locally convex space
Vector optimization

ABSTRACT

We establish a pre-order principle. From the principle, we obtain a very general set-valued Ekeland variational principle, where the objective function is a set-valued map taking values in a quasi-ordered linear space and the perturbation contains a family of set-valued maps satisfying certain property. From this general set-valued Ekeland variational principle, we deduce a number of particular versions of set-valued Ekeland variational principle, which include many known Ekeland variational principles, their improvements and some new results.

1. Introduction

In 1972, Ekeland [13] (see also [14,15]) gave a variational principle, now known as Ekeland variational principle (for short, EVP), which says that for any lower semicontinuous function f bounded from below on a complete metric space, a slightly perturbed function has a strict minimum. In the last four decades, the famous EVP emerged as one of the most important results of nonlinear analysis and it has significant applications in optimization, optimal control theory, game theory, fixed point theory, nonlinear equations, dynamical systems, etc.; see for example [3,10,14,15,21,36,51]. Motivated by this wide usefulness, many authors have been interested in extending EVP to vector-valued maps or set-valued maps with values in a vector space quasi-ordered by a convex cone, see, for example, [2,4–10,12,16,17,21–25,27–31,33–35,40–42,44,45,47,48] and the references therein.

Recently, there have been many new and interesting results of EVP for set-valued maps. Here, we only mention some results which are related to this paper. In [24], Ha introduced a strict minimizer of a set-valued map by virtue of Kuroiwa's set optimization criterion (see [32]). Using the concept of cone extensions and Dancs–Hegedus–Medvegyev theorem (see [11]) she established a new version (see [24, Theorem 3.1])

☆ This work was supported by the National Natural Science Foundation of China (10871141).
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http://dx.doi.org/10.1016/j.jmaa.2014.05.027
0022-247X/

of EVP for set-valued maps, which is expressed by the existence of a strict minimizer for a perturbed set-valued optimization problem. Inspired by Ha's work and using the Gerstewitz's function (see, for example, [18–20]), the author [41] obtained an improvement of Ha's version of EVP by relaxing several assumptions. In the above Ha's and Qiu's versions, the perturbation is given by a nonzero element k_0 of the ordering cone multiplied by the distance function $d(\cdot,\cdot)$, i.e., its form is as $d(\cdot,\cdot)k_0$ (disregarding a constant coefficient); and the objective functions are set-valued maps. Bednarczuk and Zagrodny [7] proved a vectorial EVP for a sequentially lower monotone vector-valued map (which is called a monotonically semicontinuous map in [7]), where the perturbation is given by a convex subset H of the ordering cone multiplied by the distance function $d(\cdot,\cdot)$, i.e., its form is as $d(\cdot,\cdot)H$. This generalizes the case where directions of the perturbations are singletons k_0. More generally, Gutiérrez, Jiménez and Novo [23] introduced a set-valued metric, which takes values in the set family of all subsets of the ordering cone and satisfies the triangle inequality. By using it they gave an original approach to extending the scalar-valued EVP to a vector-valued map, where the perturbation contains a set-valued metric. They also deduced several special versions of EVP involving approximate solutions for vector optimization problems and discussed their interesting applications in optimization. In the above EVPs given by Bednarczuk and Zagrodny [7] and by Gutiérrez, Jiménez and Novo [23], the objective maps are all a vector-valued (single-valued) map and the perturbations contain a convex subset of the ordering cone and a set-valued metric with values in the ordering cone, respectively.

Very recently, Liu and Ng [33], Tammer and Zălinescu [48] and Flores-Baźan, Gutiérrez and Novo [17] further considered more general versions of EVP, where not only the objective map is a set-valued map, but also the perturbation is a set-valued map, even a family of set-valued maps satisfying certain property. In particular, Liu and Ng [33] established several set-valued EVPs, where the objective map is a set-valued map and the perturbation is as the form $\gamma d(\cdot,\cdot)H$ or $\gamma' d(\cdot,\cdot)H$, $\gamma' \in (0,\gamma)$, where $\gamma > 0$ is a constant, $d(\cdot,\cdot)$ is the metric on the domain space and H is a closed convex subset of the ordering cone. Using the obtained EVPs, they provided some sufficient conditions ensuring the existence of error bounds for inequality systems. Tammer and Zălinescu [48] presented new minimal point theorems in product spaces and the corresponding set-valued EVPs. As special cases, they derived many of the previous EVPs and their extensions, for example, extensions of EVPs of Isac–Tammer's (see [28]) and Ha's versions (see [24]). Through an extension of Brézis–Browder principle, Flores-Baźan, Gutiérrez and Novo [17] established a general strong minimal point existence theorem on quasi-ordered spaces and deduced several very general set-valued EVPs, where the objective map is a set-valued map and the perturbation even involves a family of set-valued maps satisfying "triangle inequality" property. As we have seen, these general set-valued EVPs extend and improve the previous EVPs and imply many new interesting results.

On the other hand, Bao and Mordukhovich (see [4,5]) proposed the limiting monotonicity condition on objective maps and established some enhanced versions of EVP for Pareto minimizers of set-valued maps. By using minimal element theorems for product orders in locally convex spaces, Khanh and Quy [31] generalized and improved the above enhanced versions of EVP. Particularly, they extended the direction of the perturbation from a single positive vector to a convex subset of the positive cone and removed the assumption in [4,5] that the objective map is level closed.

In this paper, we first establish a pre-order principle, which consists of a pre-order set $(X,\preceq)$ and an extended real-valued function η which is monotone with respect to $\preceq$. The pre-order principle states that there exists a strong minimal point dominated by any given point provided that the monotone function η satisfies three general conditions. From the pre-order principle, we obtain a very general set-valued EVP, where the objective function is a set-valued map taking values in a quasi-ordered linear space and the perturbation contains a family of set-valued maps satisfying certain property. Our assumption is accurate and weaker than ones appeared in the previous EVPs. And our proof is clear and concise. The key to the proof is to distinguish two different points by scalarizations. From the general EVP, we can deduce all of the above mentioned set-valued EVPs, their improvements and some new versions. In particular, our pre-order principle also implies generalizations of Khanh and Quy's minimal element theorems for product