J. Math. Anal. Appl. 419 (2014) 1004–1022

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

# Local uncertainty principles for the Cohen class

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A R T I C L E I N F O

*Article history:*
Received 6 August 2013
Available online 9 May 2014
Submitted by R.H. Torres

*Keywords:*
Time–frequency representations
Wigner sesquilinear and quadratic form
Local uncertainty principles

A B S T R A C T

In this paper we analyze time–frequency representations in the Cohen class, i.e., quadratic forms expressed as a convolution between the classical Wigner transform and a kernel, with respect to uncertainty principles of local type. More precisely the results we obtain concerning the energy distribution of these representations show that a "too large" amount of energy cannot be concentrated in a "too small" set of the time–frequency plane. In particular, for a signal $f \in L^2(\mathbb{R}^d)$, the energy of a time–frequency representation contained in a measurable set $M$ must be controlled by the standard deviations of $|f|^2$ and $|\hat{f}|^2$, and by suitable quantities measuring the size of $M$.



## 1. Introduction

In this paper we prove local uncertainty principles for time–frequency representations in the Cohen class, i.e. quadratic forms of the kind

$$Q_\sigma f(x,\omega) := (\sigma * \operatorname{Wig} f)(x,\omega), \tag{1.1}$$

where $\operatorname{Wig} f$ is the classical Wigner transform, defined as

$$\operatorname{Wig} f(x,\omega) = \int e^{-2\pi i t\omega} f\left(x+\frac{t}{2}\right)\overline{f\left(x-\frac{t}{2}\right)}\,dt,$$

and $\sigma$ is a function or distribution on $\mathbb{R}^{2d}$. This class appears both as a widely used set of time–frequency representations, as well as in connection with some theoretical aspects of harmonic analysis, for example Weyl symbols of localization operators belong to this class. References can be found in [6–8,14,18,13,19].

Our purpose is to provide a reformulation, in the framework of the time–frequency representations, of the local uncertainty principle for the Fourier transform introduced by Price and Faris in [17,12,16].

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http://dx.doi.org/10.1016/j.jmaa.2014.05.002
0022-247X/

In order to motivate the main results of this paper, which are contained in Sections 2, 3, 4, we begin by reviewing some basic facts on the Cohen class; we recall then the local uncertainty principle of Price and compare it with the classical Heisenberg uncertainty principle.

The expression (1.1) makes sense for $f \in \mathcal{S}(\mathbb{R}^d)$ and $\sigma \in \mathcal{S}(\mathbb{R}^{2d})$, and in this case $Q_\sigma f \in \mathcal{S}(\mathbb{R}^{2d})$. Other more general functional frameworks are however possible in the Schwartz distribution space $\mathcal{S}'$, in such a way that the convolution in (1.1) is well defined. For example, when $f \in L^2(\mathbb{R}^d)$ and $\sigma \in L^1(\mathbb{R}^{2d})$ we obtain that $Q_\sigma f$ is a well defined element of $L^2(\mathbb{R}^{2d})$.

From the point of view of time–frequency analysis, when considering separately a function $f$ and its Fourier transform $\hat{f}$ we analyze separately the energy distribution of the "signal" $f$ with respect to time, represented by $|f(x)|^2$, and the energy distribution of $f$ with respect to frequency, represented by $|\hat{f}(\omega)|^2$. A time–frequency representation $Q_\sigma f(x,\omega)$ gives the energy distribution of a signal $f$ with respect to time $x$ and frequency $\omega$ at the same time, and in fact it doubles the dimension of its domain, being $(x,\omega) \in \mathbb{R}^{2d}$. The Cohen class contains the most important covariant representations, and moreover gives the freedom to design the kernel $\sigma$ in order that the corresponding form $Q_\sigma$ has specific features. In this framework we refer for example to [7, Chapter 11], [1,2,15]. As particular cases of the representation $Q_\sigma$ we recover the Wigner transform when $\sigma$ is the Dirac distribution $\delta$; moreover, if $\tau \in [0,1]$, $\tau \neq 1/2$, and $\sigma(x,\omega) = \frac{2^d}{|2\tau-1|^d} e^{2\pi i \frac{2}{2\tau-1} x\omega}$, the corresponding representation $Q_\sigma$ becomes the $\tau$-Wigner transform

$$\mathrm{Wig}_\tau f(x,\omega) = \int e^{-2\pi i t\omega} f(x+\tau t)\overline{f\big(x-(1-\tau)t\big)}\, dt, \tag{1.2}$$

see for example [4] (the classical Wigner transform is obtained by letting $\tau = 1/2$). For a deep investigation of the Winger representations in connection with symplectic geometry and quantization we refer to [9–11]. In the cases $\tau = 0$ and $\tau = 1$ we get the Rihaczek and conjugate Rihaczek forms, given by

$$Rf(x,\omega) = e^{-2\pi i x\omega} f(x)\overline{\hat{f}(\omega)} \quad \text{and} \quad R^* f(x,\omega) = e^{2\pi i x\omega}\overline{f(x)}\hat{f}(\omega),$$

respectively. Another relevant class of time–frequency representations contained in the Cohen class is the Spectrogram, defined as follows. Given a "window" function $\phi \in \mathcal{S}(\mathbb{R}^d)$, the Gabor transform of $f \in \mathcal{S}(\mathbb{R}^d)$ is given by $V_\phi f(x,\omega) = \int e^{-2\pi i t\omega} f(t)\overline{\phi(t-x)}\, dt$. Then, for $\phi,\psi \in \mathcal{S}(\mathbb{R}^d)$ and $f \in \mathcal{S}(\mathbb{R}^d)$ (with possible generalizations to larger functional settings) the (generalized) spectrogram is defined as follows:

$$\mathrm{Sp}_{\phi,\psi} f(x,\omega) = (V_\phi f \cdot \overline{V_\psi f})(x,\omega). \tag{1.3}$$

For $\phi = \psi$ we have in particular the classical spectrogram $|V_\phi f(x,\omega)|^2$. We refer to [3,14] for a treatment of (1.3) and for further references. Here we just recall that the generalized spectrogram belongs to the Cohen class, and the corresponding kernel is $\sigma = \mathrm{Wig}(\tilde{\psi},\tilde{\phi})$, where $\tilde{g}(t) := g(-t)$.

We shall be concerned with local uncertainty principles for representations in the Cohen class, transferring to the time–frequency frame the idea of local uncertainty principle of Price [17] for the Fourier transform. We set $\|\cdot\|_{L^p(E)}$ for the usual $L^p$-norm on $E \subset \mathbb{R}^d$ or $E \subset \mathbb{R}^{2d}$ (if $E = \mathbb{R}^d$ or $E = \mathbb{R}^{2d}$ we simply write $\|\cdot\|_p$). Moreover, the Fourier transform of $f \in \mathcal{S}(\mathbb{R}^d)$ is given by $\hat{f}(\omega) = \int e^{-2\pi i t\omega} f(t)\, dt$, with standard extensions to $L^2(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$.

For a function $f \in L^2(\mathbb{R})$ (for simplicity we consider here $d = 1$) we define $\bar{x} = \int x|f(x)|^2\, dx$ and $\bar{\omega} = \int \omega|\hat{f}(\omega)|^2\, d\omega$. Then the corresponding standard deviations are $\sigma_f = \|(x-\bar{x})f(x)\|_2$ and $\sigma_{\hat{f}} = \|(\omega-\bar{\omega})\hat{f}(\omega)\|_2$. The classical Heisenberg uncertainty principle states that, for every $f \in L^2(\mathbb{R})$ with $\|f\|_2 = 1$ we have

$$\sigma_f \sigma_{\hat{f}} \geq \frac{1}{4\pi}. \tag{1.4}$$