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Gevrey asymptotics of series in Mourtada-type compensators used for linearization of an analytic $1 : -1$ resonant saddle

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ABSTRACT

Given a $1 : -1$ resonant saddle singularity of a planar analytic vector field, we provide a linearization procedure using a series expansion in compensators of Mourtada-type, and show that this series has Gevrey-1 asymptotics. In case of an analytic Poincaré–Dulac normal form we show that this transformation is analytic as a function of the compensators.

1. Introduction, background and statement of the result

We consider a $1 : -1$ saddle type singularity at $(0, 0)$ of a planar vector field X, that is: the eigenvalues have a ratio $1 : -1$. For the simplicity of the exposition we will take the eigenvalues to be 1 and -1, although this is not essential.

We are interested in transformations, near $(0, 0)$, conjugating X to the linear saddle $x\partial/\partial x - y\partial/\partial y$. It is well known [9] that such a conjugacy can always be obtained by a C^1 transformation (surely, a C^2 linearization is in general not possible due to the $1 : -1$ resonance of the eigenvalues). Such a transformation is usually obtained, roughly speaking, from a fixed point contraction argument in some function space, so it is not very explicit. There is no further information about the transformation itself.

When starting from an analytic X we throw away a lot of information in this way. In this paper we intend to provide linearizing transformations, tangent to the identity, taking the analyticity into account as much as possible.

Let us first recall some results from normal form theory. It is well known [4] that the formal normal form of X (i.e. using formal power series in (x, y)) is of the form

$$x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + x.F(xy)\frac{\partial}{\partial x} - y.G(xy)\frac{\partial}{\partial y} \tag{1}$$

where F and G are formal power series. Let us abbreviate $u = xy$.

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On the formal level, it is generically possible [12,19,10] to further simplify this expression to a polynomial normal form

$$x\left(1+Au^N\right).P(u)\frac{\partial}{\partial x}-y\left(1+Bu^N+Cu^{2N}\right).P(u)\frac{\partial}{\partial y} \tag{2}$$

where P is a polynomial of degree at most N with $P(0)=1$ and $A\neq B$. For an extensive overview of related normal form theory we refer to [13] and the references therein.

If X is C^∞ then there is a C^∞ transformation into (1) or (2); if X is analytic (even Gevrey-1) the conjugacy with (1) can be taken formally Gevrey-1 [1].

On the other hand, we can use variables that take resonances into account, and try to get more control. The idea to use these kind of variables, sometimes called compensators, near a resonant singular point of saddle type, was already used in [7,16,15]. They appear naturally when calculating the Dulac transition map near the origin; for a geometric description, and extension in case the resonance is unfolded, we refer to [16]. Hence it can be expected that these compensators can be useful in the construction of conjugacies to normal form. This is the approach that we will make in this paper. We will use a series expansion in these variables and show that it has Gevrey-1 asymptotics provided that X is analytic.

In order to fix the ideas of the reader, let us give a simple example: consider the vector field

$$X(x,y)=x(1+u)\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}. \tag{3}$$

In this particular case there is a first integral $H(x,y)=xy/(1-xy\log|y|)$ and we can linearize this as follows. Put $\tilde{x}=x/(1-xy\log|y|)$, then $(\tilde{x}y)'=0$, and hence $\tilde{x}'=\tilde{x}$. This shows that the change of variables $(x,y)\mapsto(\tilde{x},y)$ conjugates X to $\tilde{x}\partial/\partial\tilde{x}-y\partial/\partial y$. Notice that the variable $y\log|y|$ is still a 'small' variable when y is small. Moreover, in this case we see that the conjugating transformation is analytic in the variables $(x,y\log|y|)$, although it is not C^1 in (x,y).

Because of the fact that, in general, we can only expect a Gevrey-1 conjugacy between X and its formal normal form, it would perhaps be more natural to start from a Gevrey-1 vector field X, and to assume that it is in formal normal form. However, for the estimates of the asymptotics of the series in the compensators in Section 5 we use the analyticity of X. For the moment we have no idea whether the methods can be adapted to the Gevrey-class. This is the reason why we will start, in Section 2, from an analytic vector field in a general form i.e. where no normal form reduction is performed, except for 'straightening out' stable and unstable manifold, see Eq. (6) below.

Let us come to the main result. A formal series $\sum_n \alpha_n z^n$ in one variable z is called Gevrey-1 if the growth of the coefficients α_n can be estimated by $|\alpha_n|\leq Kn!L^n$ for some constants $K,L\geq 0$. In the statement of the next theorem we will have estimates on power series in several variables, which behave Gevrey-1-like in some variables and which are 'analytic' (i.e. geometric growth of the coefficients) in some other variables. There seems to be no consensus about the definition of Gevrey series in several variables, but for our purposes the definition is given from the estimate (5) below.

Theorem 1. *Let X be a planar analytic vector field near the singularity $(0,0)$ with eigenvalues 1 and -1. We may, and do, assume that the manifolds $x=0$ and $y=0$ are invariant. There exists a formally linearizing change of variables $(x,y)=(\tilde{x}(1+\varphi_1(\tilde{x},\tilde{y})),\tilde{y}(1+\varphi_2(\tilde{x},\tilde{y})))$ where the φ_I, $I=1,2$, are formal series in $(\tilde{x},\tilde{y})$ and $(\tilde{x}\log|\tilde{x}|,\tilde{y}\log|\tilde{y}|)$* ('the compensators') *of the form*

$$\varphi_I=\sum_{i,j,k,l}\alpha^I_{(i,j,k,l)}\tilde{x}^i\tilde{y}^j\left(\tilde{x}\log|\tilde{x}|\right)^k\left(\tilde{y}\log|\tilde{y}|\right)^l \tag{4}$$

where we have the following Gevrey-type estimate: