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Higher-order convolutions for Bernoulli and Euler polynomials ☆

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ABSTRACT

We prove convolution identities of arbitrary orders for Bernoulli and Euler polynomials, i.e., sums of products of a fixed but arbitrary number of these polynomials. They differ from the more usual convolutions found in the literature by not having multinomial coefficients as factors. This generalizes a special type of convolution identity for Bernoulli numbers which was first discovered by Yu. Matiyasevich.

1. Introduction

The Bernoulli and Euler numbers, as well as their polynomial analogues, have numerous important applications in number theory, combinatorics, numerical analysis, and other areas of pure and applied mathematics. They have therefore been studied extensively over the last two centuries. For the most important properties see, for instance, [1, Ch. 23] or its successor [14, Ch. 24]. Other good references are [9], [10], or [13]. For a general bibliography, see [7].

The Bernoulli numbers B_n, $n = 0, 1, 2, \ldots$, can be defined by the exponential generating function

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \quad \bigl(|t| < 2\pi\bigr). \tag{1.1}$$

The first few values of the Bernoulli numbers are listed in Table 1. A large number of linear and nonlinear recurrence relations for these numbers are known, and such relations also exist for the Bernoulli *polynomials*

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Table 1
$B_n, E_n, G_n, B_n(x)$ and $E_n(x)$ for $0 \le k \le 6$.

n	B_n	E_n	G_n	$B_n(x)$	$E_n(x)$
0	1	1	0	1	1
1	$-1/2$	0	1	$x - \frac{1}{2}$	$x - \frac{1}{2}$
2	$1/6$	-1	-1	$x^2 - x + \frac{1}{6}$	$x^2 - x$
3	0	0	0	$x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$	$x^3 - \frac{3}{2}x^2 + \frac{1}{4}$
4	$-1/30$	5	1	$x^4 - 2x^3 + x^2 - \frac{1}{30}$	$x^4 - 2x^3 + x$
5	0	0	0	$x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x$	$x^5 - \frac{5}{2}x^4 + \frac{5}{2}x^2 - \frac{1}{2}$
6	$1/42$	-61	-3	$x^6 - 3x^5 + \frac{5}{2}x^4 - \frac{1}{2}x^2 + \frac{1}{42}$	$x^6 - 3x^5 + 5x^3 - 3x$

and for Euler numbers and polynomials which are defined at the beginning of Section 2. This paper deals with *nonlinear* recurrence relations, namely convolution identities for Bernoulli and Euler numbers and polynomials. The most basic one is

$$\sum_{j=0}^{n} \binom{n}{j} B_j B_{n-j} = -nB_{n-1} - (n-1)B_n \quad (n \ge 1), \tag{1.2}$$

which is due to Euler. For brief historical discussions of linear and nonlinear recurrence relations, with numerous references, see [4] and [3], respectively.

More recently, two different types of convolution identities were discovered by Miki and Matiyasevich, respectively, and subsequently generalized and extended by other authors. Miki's identity [12] is

$$\sum_{j=2}^{n-2} \frac{B_j B_{n-j}}{j(n-j)} - \sum_{j=2}^{n-2} \binom{n}{j} \frac{B_j B_{n-j}}{j(n-j)} = 2H_n \frac{B_n}{n} \quad (n \ge 4), \tag{1.3}$$

where H_n is the nth harmonic number defined by $H_n := 1 + \frac{1}{2} + \cdots + \frac{1}{n}$. The second identity was discovered experimentally by Matiyasevich (Identity #0202 in [11]) and was later proved by other authors (see [2] for references). It can be written in an equivalent form as

$$(n+2) \sum_{j=2}^{n-2} B_j B_{n-j} - 2 \sum_{j=2}^{n-2} \binom{n+2}{j} B_j B_{n-j} = n(n+1)B_n \quad (n \ge 4). \tag{1.4}$$

The remarkable feature of both identities is the fact that they combine two different kinds of convolutions.

Subsequently, the identities (1.3) and (1.4) were extended to Bernoulli *polynomials* by Gessel [8] and by Pan and Sun [15], respectively. In the same paper [8], Gessel also extended (1.3) to third-order convolutions, i.e., sums of products of three Bernoulli numbers, and indicated that his method could provide convolutions of any order.

Later the first author [2] found different and simpler proofs of the polynomial analogues of (1.3) and (1.4). He also proved numerous other "Miki-type" and "Matiyasevich-type" identities involving Bernoulli, Euler, and Genocchi numbers and polynomials.

It is the purpose of this paper to prove Bernoulli polynomial analogues of arbitrary order of Matiyasevich's identity (1.4) and to do the same for Euler polynomials. The results and some corollaries are stated in Section 2, along with some required definitions. Section 3 contains two important lemmas, and the proofs of the theorems are then given in Sections 4 and 5. We conclude the paper with some additional remarks in Section 6.