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# Convergence of partial maps

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ABSTRACT

Given metric spaces $(X, d)$ and $(Y, \rho)$, a partial map between $X$ and $Y$ is a pair $(D, u)$, where $D$ is a closed subset of $X$ and $u : D \to Y$ is a function. We introduce a general convergence notion for nets of such partial functions. While our initial description is variational in nature, we show that this description amounts to bornological convergence of the associated net of graphs as defined by Lechicki, Levi and Spakowski [26] with respect to a natural bornology on $X \times Y$, and which places the work on continuous partial functions of Brandi, Ceppitelli, and Holá [12,13,20,21] in a general framework.



## 1. Introduction

Let $(X, d)$ and $(Y, \rho)$ be metric spaces. By a *partial function* or a *partial map* from $X$ to $Y$, we mean a pair $(D, u)$ where $D$ is a nonempty closed subset of $X$ and $u : D \to Y$ is a function (not assumed continuous on $D$). We denote the space of all such maps by $\mathcal{P}[X, Y]$, while by $\mathcal{C}[X, Y]$ we mean those partial maps that are continuous on their respective domains.

Continuous partial functions were first considered by Kuratowski [24] where $X$ was assumed compact so that for each $(D, u) \in \mathcal{C}[X, Y]$, the graph of $u$, which we denote by $\mathrm{Gr}(u)$, is compact as well. He topologized $\mathcal{C}[X, Y]$ by equipping the graph space with the classical Hausdorff metric topology [3], identifying partial functions with their graphs.

In the more general setting we have introduced, graphs of continuous partial functions, while not compact, will be closed subsets of $X \times Y$. Partial functions play a central role in mathematical economics, since they are typical utility functions for agents. Tastes of agents on a space $X$ are usually represented by a preference relation on $X$, that is, a subset $R$ of $X \times X$ where $(x, y) \in R$ means the agent prefers alternative $x$

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to $y$. Following the seminal work of Debreu [15], utility functions have been considered as more suitable mathematical tools to represent agents' preferences. Similarities of agents can be described by a convergence or topology on partial maps.

Having a dual representation of preferences – relations on one side equipped themselves with a convergence or topology and functions on the other – demands that one connects the two possible conceptions of similarity. This actually was the main motivation for Back to introduce the generalized compact open topology on utility functions [2]. By exploiting a result of Levin [27], Back showed that when $X$ is locally compact and separable, classical Kuratowski convergence of preference relations can be expressed in terms of the convergence of suitable utility functions representing the preferences in his topology. Our motivation here is to give a new variational definition of convergence of partial maps that is compatible with Back's topology in the locally compact setting and that might be applicable beyond.

This paper proposes a new convergence on the set of the partial maps that can be described in different ways. While our initial description is variational in nature, it can also be described in terms of convergence of graphs and thus is consonant with the initial paper of Kuratowski. All of our descriptions involve bornologies, macroscopic structures employed over the last 25 years to describe convergence of nets or sequences of sets, the prototype being the now classical Attouch–Wets convergence, sometimes called bounded Hausdorff convergence (see, e.g., [1,3,7,28]).

**Definition 1.1.** A *bornology* $\mathcal{B}$ on a metric space $(X, d)$ is a family of nonempty subsets of $X$ covering $X$, that is stable under taking finite unions, and that is hereditary, i.e., stable under taking nonempty subsets.

The smallest bornology on $X$ is the family of nonempty finite subsets of $X$, $\mathcal{F}$, and the largest is the family of all nonempty subsets of $X$, $\mathcal{P}_0(X)$. Other important bornologies are: the family $\mathcal{B}_d$ of the nonempty $d$-bounded subsets; the family $\mathcal{B}_{tb}$ of the nonempty $d$-totally bounded subsets; and the family $\mathcal{K}$ of nonempty subsets of $X$ with compact closure. Bornologies in general topology were first considered by Hu [22]; their role in locally convex spaces is the subject of the monograph of Hogbe-Nlend [19].

Given a bornology $\mathcal{B}$ on $(X, d)$, we can describe an associated convergence notion $\mathcal{P}(\mathcal{B})$ on $\mathcal{P}[X, Y]$: a rule that assigns to each net in $\mathcal{P}[X, Y]$ a (potentially empty) set of limits in $\mathcal{P}[X, Y]$ (for adequate information on nets for our purposes, see [23]). First, given a nonempty subset $A$ of a metric space and $\epsilon > 0$, let $A^\epsilon$ denote the $\epsilon$-*enlargement* of $A$, that is, the union of all open balls of radius $\epsilon$ whose centers run over $A$.

**Definition 1.2.** Let $(X, d)$, $(Y, \rho)$ be metric spaces, and let $\mathcal{B}$ be a bornology on $X$. Let $\Gamma$ be a directed set and let $\langle(D_\gamma, u_\gamma)\rangle_{\gamma\in\Gamma}$ be a net in $\mathcal{P}[X, Y]$. We say that the net is $\mathcal{P}(\mathcal{B})$-*convergent* to $(D, u)$, and write $(D, u) \in \mathcal{P}(\mathcal{B})\text{-}\lim(D_\gamma, u_\gamma)$, if for every $B \in \mathcal{B}$ and $\epsilon > 0$, there exists $\gamma_0 \in \Gamma$ such that the following two conditions hold for all indices $\gamma \geq \gamma_0$:

(1) for each nonempty subset $B_1$ of $B$, $u(D \cap B_1) \subset [u_\gamma(D_\gamma \cap B_1^\epsilon)]^\epsilon$;
(2) for each nonempty subset $B_1$ of $B$, $u_\gamma(D_\gamma \cap B_1) \subset [u(D \cap B_1^\epsilon)]^\epsilon$.

Notice that in conditions (1) and (2), the inside enlargement in the last expression is taken in $X$ while the outside enlargement is taken in $Y$. Even within $\mathcal{C}[X, Y]$, limits need not be unique and we will characterize bornologies for which uniqueness of limits occurs.

We will present analytical alternatives for conditions (1) and (2) that will be more manageable to check convergence in practice. But the most tangible and visual description of $\mathcal{P}(\mathcal{B})$-convergence is the following: for each $B \in \mathcal{B}$ and $\epsilon > 0$, eventually both $\mathrm{Gr}(u_\gamma) \cap (B \times Y) \subset \mathrm{Gr}(u)^\epsilon$ and $\mathrm{Gr}(u) \cap (B \times Y) \subset \mathrm{Gr}(u_\gamma)^\epsilon$. In this formulation, the enlargement is taken with respect to any metric compatible with the product uniformity. For definiteness, we choose the *box metric* defined by $(d \times \rho)((x_1, y_1), (x_2, y_2)) := \max\{d(x_1, x_2), \rho(y_1, y_2)\}$. Readers familiar with the set convergence literature will immediately recognize this as bornological convergence