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Study of a nonlinear Kirchhoff equation with non-homogeneous material

Giovany M. Figueiredo [a], Cristian Morales-Rodrigo [b,*], João R. Santos Júnior [a], Antonio Suárez [b]

[a] *Universidade Federal do Pará, Faculdade de Matemática, CEP: 66075-110 Belém, Pa, Brazil*
[b] *Dpto. de Ecuaciones Diferenciales y Análisis Numérico, Fac. de Matemáticas, Univ. de Sevilla, C/. Tarfia s/n, 41012, Sevilla, Spain*

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ABSTRACT

In this paper we study a non-homogeneous elliptic Kirchhoff equation with nonlinear reaction term. We analyze the existence and uniqueness of positive solution. The main novelty is the inclusion of non-homogeneous term making the problem without a variational structure. We use mainly bifurcation arguments to get the results.

1. Introduction

In this paper we study the following nonlinear Kirchhoff equation with non-homogeneous material

$$\begin{cases} -M\big(x, \|u\|^2\big)\Delta u = \lambda u^q & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geqslant 1$, is a bounded and regular domain, $0 < q \leqslant 1$, $\lambda \in \mathbb{R}$ and

$$M(x,s) := a(x) + b(x)s, \qquad \|u\|^2 = \int_\Omega |\nabla u|^2\,dx,$$

with $a, b \in C^\gamma(\overline{\Omega})$, $\gamma \in (0,1)$ and $a(x) \geqslant a_0 > 0$, $b \geqslant 0$. Eq. (1.1) is the steady-state problem associated to the time dependent problem

* Corresponding author.
E-mail addresses: giovany@ufpa.br (G.M. Figueiredo), cristianm@us.es (C. Morales-Rodrigo), joaojunior@ufpa.br (J.R. Santos Júnior), suarez@us.es (A. Suárez).

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$$\begin{cases} u_{tt} - M\big(x, \|u\|^2\big)\Delta u = f & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } \Omega, \end{cases} \tag{1.2}$$

which models small vertical vibrations of an elastic string with fixed ends when the density of the material is not constant. The problem (1.2) was proposed by J.L. Lions [16] (see also [15] and [23]). The elliptic version of (1.2) was studied in [21] and [25] for bounded and unbounded domains, respectively. In these papers a fixed point argument and the Galerkin method are used to prove the existence of a solution.

However, in contrast with the non-homogeneous case, when a and b are positive constants the problem has a variational structure and has been investigated extensively during last years. In [2,7,8,10,14,18,24] the following problem was studied

$$\begin{cases} -M\big(\|u\|^2\big)\Delta u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

for different properties on f. In [2] and [18] the Mountain Pass Theorem and a truncation argument are applied to prove the existence of a solution when f is subcritical (see also [8] for the critical case). In [7] the Mountain Pass Theorem and the Ekeland Principle were used to show the existence of multiple non-trivial solutions of (1.3) with a concave nonlinearity. In [24] variational results were employed for nonlinearities f which are resonant at an eigenvalue. In [14] existence of positive solutions was showed using topological degree arguments and variational method for functions f asymptotically linear at zero and asymptotically 3-linear at infinity. When $\Omega = \mathbb{R}^N$ the problem

$$\begin{cases} M\Big(\displaystyle\int_{\mathbb{R}^N} |\nabla u|^2\,dx + V(x)u\Big)\big[-\Delta u + V(x)u\big] = f(u) & \text{in } \mathbb{R}^N, \\ u \in H^1\big(\mathbb{R}^N\big), \end{cases} \tag{1.4}$$

has been analyzed under appropriate assumptions on V and f. In [1] it is shown existence of solution for f subcritical and critical. Multiplicity of solutions was showed in [10,13,18,27,26,28] using genus or category theory. The case in which the Laplace operator is replaced by the p-Laplacian or the $p(x)$-Laplacian has been considered in [6] and [5] respectively. The case where M is the identity and $V(x) = b > 0$ is studied in [4] via minimization and in [12] by a monotonicity trick. For sign changing solutions see the papers [20,19,29].

The purpose of this paper is to take a first step to study the problem

$$-M\big(x, \|u\|^2\big)\Delta u = f(x,u)$$

for general non-linearities f. For that, we have chosen the sublinear case $f(x,s) = \lambda s^q$ for $0 < q \leqslant 1$. We employ two different techniques to study our problem. For the case $q < 1$ we use the bifurcation method to show that there exists a positive solution for all $\lambda > 0$ and no positive solution for $\lambda \leqslant 0$. This result is similar to the one obtained in the homogeneous case, although the techniques that we apply are different. In order to apply the bifurcation method, we need to rewrite (1.1) as a fixed point equation of a compact operator. For that, we have to deal with the case $f = f(x)$.

For the case $q = 1$ we have used an argument based on the eigenvalue problems of elliptic equations and their properties. In this case, we can see the consequences of the non-homogeneous term b. In this case we show the existence of positive solution for $\lambda \in (\lambda_0, \lambda_1)$ where λ_0 and λ_1 are positive constants; while for the homogeneous case the existence holds for $\lambda > \lambda_0$, see Section 4 for details.

In all the cases (lineal, $q = 1$ and $q < 1$) we have proved the uniqueness of positive solution of (1.1). In our knowledge, the results are new even in the homogeneous case.