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On the determination of a function from an elliptical Radon transform

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ABSTRACT

In recent years, Radon-type transforms that integrate functions over various sets of ellipses/ellipsoids have been considered in synthetic aperture radar, ultrasound reflection tomography, and radio tomography. In this paper, we consider the transform that integrates a given function in $\mathbb{R}^n$ over a set of solid ellipses (when $n = 2$) or solid ellipsoids of rotation (when $n \geqslant 3$) with a fixed eccentricity and foci restricted to a hyperplane. Inversion formulas are obtained for appropriate classes of functions that are even with respect to the hyperplane. Stability estimates and local uniqueness results are also provided.

1. Introduction

Radon-type transforms that integrate functions over various sets of ellipses/ellipsoids have been arising in the recent decade, due to studies in Synthetic Aperture Radar (SAR) [2,4,8,9], Ultrasound Reflection Tomography (URT) [7,1], and radio tomography [15–17]. In particular, radio tomography is a new imaging method, which uses a wireless network of radio transmitters and receivers to image the distribution of attenuation within the network. The usage of radio frequencies brings in significant non-line-of-sight propagation, since waves propagate along many paths from a transmitter to a receiver. Given a transmitter and a receiver, wave paths observed for a given duration are all contained in an ellipsoid with foci at these two devices. It was thus suggested in [15–17] to approximate the obtained signal by the volume integral of the attenuation over this ellipsoid, which is the model we study in this article.

Due to these applications, there have been several papers devoted to such "elliptical Radon transform." The family of ellipses with one focus fixed at the origin and the other one moving along a given line was considered in [9]. In the same paper, the family of ellipses with a fixed focal distance was also studied. The authors of [7,1] dealt with the case of circular acquisition, when the two foci of ellipses with a given focal distance are located on a given circle. A family of ellipses with two moving foci was also handled in [4].

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In all these works, however, the ellipses have varying eccentricity, the ratio of the major axis to the focal distance. Also, their data were the line integrals of the function over ellipses rather than area integrals. The radio tomography application makes it interesting to consider integrals over solid ellipsoids. In this article, we consider the volume integrals of an unknown attenuation function over the family of ellipsoids of rotation in $\mathbb{R}^n$ with a fixed eccentricity and two foci located in a given hyperplane. We thus reserve the name **elliptical Radon transform** $R_E f$ for the volume integral of a function f over this family of ellipsoids. (When $n = 2$, the elliptical Radon transform $R_E f$ is the area integrals of a function over the family of ellipses with a fixed eccentricity and two foci located in a line.)

The volume integral of a function $f(x)$ over an ellipsoid (or an ellipse for $n = 2$) of the described type is equal to zero if the function is odd with respect to the chosen hyperplane. If the hyperplane is given by $x_n = 0$, we thus assume the function $f : \mathbb{R}^n \to \mathbb{R}$ to be even with respect to x_n: $f(x', x_n) = f(x', -x_n)$ where $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$.

Given a Radon-type transform, one is usually interested, among others, in the following questions: inversion formulas, uniqueness for a local data problem, and a stability estimate [10,11]. These are the issues we address below.

The problem is stated precisely in Section 2. Two inversion formulas are presented in Sections 3 and 4. Analogue of the Fourier slice theorem is obtained in Section 3 by taking the Fourier transform with respect to the center and a radial Fourier transform with respect to the half distance between two foci. This theorem plays a critical role in getting a stability estimate. The formula discussed in Section 4 is obtained by taking a Fourier type transform and needs less integration than the previous one in Section 3. A stability estimate is handled in Section 5. Section 6 is devoted to uniqueness for a local data problem.

2. Formulation of the problem

We consider all solid ellipses (when $n = 2$) or solid ellipsoids of rotation (when $n \geqslant 3$) in $\mathbb{R}^n$ with a fixed eccentricity $1/\lambda$, where $\lambda > 1$ and foci located in the hyperplane $x_n = 0$. We will identify this hyperplane with $\mathbb{R}^{n-1}$. While a function f depends on n parameters, the set of such ellipsoids depends on $2n - 2$ parameters. This means that when $n \geqslant 3$, the problem of inverting the elliptical Radon transform is $n - 2$-dimensions overdetermined. To reduce the overdeterminacy, we require that the focal axis is parallel to a given line, for instance, the x_1 coordinate axis. When $n = 2$, i.e., integral domain is an ellipse, there is no overdeterminacy and the focal axis is automatically parallel because the hyperplane is the line.

Let $u \in \mathbb{R}^{n-1}$ be the center of such an ellipse/ellipsoid and let $t > 0$ be the half of the focal distance. We denote this ellipse/ellipsoid by $E_{u,t}$. Then, the foci are

$$c_1 = (u_1 + t, u_2, \dots, u_{n-1}, 0) \quad \text{and} \quad c_2 = (u_1 - t, u_2, \dots, u_{n-1}, 0)$$

and the points $x \in E_{u,t}$ are described as follows:

$$\frac{(x_1 - u_1)^2}{\lambda^2} + \frac{(x_2 - u_2)^2}{\lambda^2 - 1} + \cdots + \frac{x_n^2}{\lambda^2 - 1} \leqslant t^2.$$

To shorten the formulas, we are going to use the following notation:

$$\nu := \sqrt{\lambda^2 - 1}.$$

The elliptical Radon transform R_E maps a locally integrable function $f(x)$ into its integrals over the solid ellipses/ellipsoids $E_{u,t}$ for all $u \in \mathbb{R}^{n-1}$ and $t > 0$:

$$R_E f(u,t) = \int\limits_{E_{u,t}} f(x)\, dx.$$

Our goals are to reconstruct f from $R_E f$ and to study properties of this transform.