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On one-parameter semigroups generated by commuting continuous injections

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ABSTRACT

The problem of the embeddability of two commuting continuous injections $f, g : I = (0, b] \to I$ in Abelian semigroups is discussed. We consider the case when there is no iteration semigroup in which f and g can be embedded. Explaining this phenomenon we modify the definition of an iteration semigroup introducing a new notion – a T-iteration semigroup of f and g, that is a family $\{f^t : I \to I,\ t \in T\}$ of continuous injections for which $f^u \circ f^v = f^{u+v}$, $u, v \in T$, such that $f = f^1$ and $g = f^s$ for an $s \in T$ and $s \notin \mathbb{Q}$, where $T \subsetneq \mathbb{R}^+$ is a dense semigroup which can be extended to a group. We determine a maximal semigroup of indices $\mathrm{Sem}(f, g) \subsetneq \mathbb{R}^+$ such that for every T-iteration semigroup $T \subset \mathrm{Sem}(f, g)$. We give also a construction of maximal T-iteration semigroups of f and g that is such semigroups for which $T = \mathrm{Sem}(f, g)$. We examine also some other Abelian semigroups of continuous functions containing f and g.

1. Introduction

We consider the problem of the embeddability of two commuting continuous injections in semi-flows called here iteration semigroups. The characterization of the embeddability of continuous commutable bijections in iteration groups is given in [12]. It turns out that, except some very regular particular case, omitting the assumption of surjectivity results in the lack of the embeddability in an iteration semigroup. In this paper such a case is considered. We explain this phenomenon and we construct the Abelian semigroups of mappings defined in the same interval I which substitute the iteration semigroups. The construction of the maximal Abelian subsemigroups containing settled two commuting mappings is given. To this end let us introduce the following notions.

Let I be an interval and let T be an additive dense subsemigroup of $\mathbb{R}^+$ such that $1 \in T$. A one parameter family $\mathcal{F} := \{f^t : I \to I,\ t \in T\}$ of continuous functions f^t such that $f^t \circ f^s = f^{t+s}$ for all $t, s \in T$ is said

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to be a *T-iteration semigroup*, however the semigroup T will be called a *support of* $\mathcal{F}$. We will also say that $\mathcal{F}$ is *supported by semigroup* T. Note that every T-iteration semigroup is Abelian.

If for every $x \in I$ the mapping $t \in T \to f^t(x)$ is an injection then a T-iteration semigroup is said to be *injective*. If $T = \mathbb{R}^+$ then T-iteration semigroup is said to be an *iteration semigroup* also called in the literature a *semiflow* (see [8]). Note that if f^1 is an injection then the remaining f^t are also injective.

If $f, g : I \to I$ are given functions and there exists a T-iteration semigroup $\{f^t : I \to I,\ t \in T\}$ such that $f^1 = f$ and $f^s = g$ for an $s \in T$ then we say that f and g are *T-embeddable*. If $T = \mathbb{R}^+$ then we will say shortly that f and g are *embeddable*.

A family of functions $\mathcal{A}$ is said to be *disjoint* whenever $f, g \in \mathcal{A}$ and $f(x) = g(x)$ for some x then $f = g$ (see [2]). Note that a T-iteration semigroup $\{f^t : I \to I,\ t \in T\}$ is disjoint if and only if, for every $t \in T$, f^t either has no fixed points or is the identity.

Denote here by $\mathbb{N}$ the set of natural numbers with 0. The mappings $f, g : I \to I$ are said to be *iteratively incommensurable* when for every $x \in I$ and every $n, m \in \mathbb{N}$ such that $n + m \neq 0$, $f^n(x) \neq g^m(x)$. In such a case the graphs of iterates are disjoint.

2. Preliminaries

Let $I = (0, b]$ be an interval. On given functions f and g we assume the general hypothesis:

(H) $f, g : I \to I$ are continuous, strictly increasing, $f \circ g = g \circ f$ and f, g are iteratively incommensurable.

Note that the assumption $I = (0, b]$ implies that f and g are not surjections, $f < id$ and $g < id$.

It is easily visible that for every $x \in I$ there exists a unique sequence $\{m_k(x)\}$ of positive integers such that $f^{m_k(x)+1}(x) \leqslant g^k(x) < f^{m_k(x)}(x)$. Moreover, there exists the finite limit

$$\lim_{k\to\infty} \frac{m_k(x)}{k} =: s(f, g),$$

and this limit does not depend on x (see [11]). This limit $s := s(f, g)$ is said to be the *iterative index of f and g*. Index $s \notin \mathbb{Q}$ if and only if f and g are iteratively incommensurable.

Assume that f and g satisfy (H). Define

$$\mathcal{N}_+(x) := \{(n, m) \in \mathbb{N} \times \mathbb{N}:\ f^n(x) \in g^m[I]\},$$
$$\mathcal{N}_-(x) := \{(n, m) \in \mathbb{N} \times \mathbb{N}:\ g^m(x) \in f^n[I]\}$$

and

$$C_+(x) := \{g^{-m} \circ f^n(x):\ (n, m) \in \mathcal{N}_+(x)\},$$
$$C_-(x) := \{f^{-n} \circ g^m(x):\ (n, m) \in \mathcal{N}_-(x)\}.$$

Put

$$L_{f,g} := C_+(x)^d,$$

where A^d means the set of all limit points of A. After [6] we quote

Proposition 1. *(See Theorem 1 in [6], Theorem 1 in [9].) The set $L_{f,g}$ does not depend on the choice of x. $C_-(x)^d = C_+(x)^d$ and $L_{f,g}$ is either a Cantor set, i.e. a perfect and nowhere dense set or $L_{f,g}$ is an interval.*