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Metaplectic group, symplectic Cayley transform, and fractional Fourier transforms

Maurice A. de Gosson [a,*], Franz Luef [b]

[a] *University of Vienna, Faculty of Mathematics (NuHAG), 1090 Vienna, Austria*
[b] *Department of Mathematics, Norwegian University of Science and Technology, 7491 Trondheim, Norway*

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ABSTRACT

We begin with a survey of the standard theory of the metaplectic group with some emphasis on the associated notion of Maslov index. We thereafter introduce the Cayley transform for symplectic matrices, which allows us to study in detail the spreading functions of metaplectic operators, and to prove that they are basically quadratic chirps. As a non-trivial application we give new formulae for the fractional Fourier transform in arbitrary dimension. We also study the regularity of the solutions to the Schrödinger equation in the Feichtinger algebra.

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* Corresponding author.

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1. Introduction

The metaplectic group $\mathrm{Mp}(d)$ has a long and rich history; its definition goes back to I. Segal [20] and D. Shale [21], and its study has been taken up from an abstract viewpoint by A. Weil [23] in connection with C. Siegel's work in number theory. Its theory has subsequently been developed and used by many authors working in various areas of mathematics and physics (see [13,18] for applications to quantum mechanics); a non-exhaustive list of contributions is (in alphabetical order) [4,6,11,17,22], and the references therein. The metaplectic group also intervenes in time-frequency analysis, especially in connection with the theory of Gabor frames (see in particular K.H. Gröchenig's treatise [14] for applications to Gabor analysis).

The aim of this article is to study the metaplectic representation from the point of view of the Weyl calculus of pseudodifferential operators, and to apply our results to a systematic treatment of the fractional Fourier transform. Some of our results have been exposed in [12,13] with quantum-mechanical applications in mind (the semiclassical Gutzwiller formula), but in a different spirit and with different notation. The exposition we give in this paper is conceptually simpler, thanks to a more systematic use of the symplectic Cayley transform, which is an object of intrinsic genuine mathematical interest. We have carefully avoided sloppy statements about the phase factors intervening in the expression of the involved metaplectic operators; while it does not matter what these factors are as long as these operators appear in conjugation formulae (*e.g.* various symplectic covariance formulae in Weyl calculus), this lack of rigor is completely fatal in many other circumstances, for instance in quantum mechanics where one considers sums of terms involving each a different operator.

At the most elementary level the metaplectic group explains the connection between two very classical notions, the Fourier transform and the symplectic group: to the Fourier transform

$$\mathcal{F}f(\omega) = \int_{\mathbb{R}^d} e^{-2\pi i\omega\cdot x} f(x)\, dx$$

(or rather a slight variant thereof, see formula (5)) the metaplectic representation associates the symplectic rotation

$$\mathcal{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

At a mathematically more sophisticated level, the metaplectic group $\mathrm{Mp}(d)$ can be defined in at least two different ways, both making (implicitly or explicitly) use of the following basic fact: the symplectic group $\mathrm{Sp}(d)$ is connected and $\pi_1[\mathrm{Sp}(d)]$ is isomorphic to the integer group $(\mathbb{Z},+)$; from this follows that $\mathrm{Sp}(d)$ has (connected) covering groups $\mathrm{Sp}_q(d)$ of all orders $q = 2, 3, \ldots, +\infty$. Having this in mind one can proceed as follows:

- Either one uses representation theory: let ρ be the Schrödinger representation of the Heisenberg group $\mathbb{H}(d)$ [8,13,14]; to every $(z,\tau) = (x,\omega,\tau) \in \mathbb{H}(d)$ it associates the unitary operator

$$\rho(z,\tau) = e^{2\pi i\tau} e^{\pi i x\cdot\omega} T_x M_\omega$$

where T_x and M_ω are, respectively, the time-shift and modulation operators; a theorem basically due to Shale and Weil [21,23] then asserts the existence of a unitary representation $[\mu]$ on $L^2(\mathbb{R}^d)$ of $\mathrm{Sp}_2(d)$ such that

$$[\mu]\bigl([\mathcal{A}]\bigr)\rho(z,\tau)[\mu]\bigl([\mathcal{A}]\bigr)^{-1} = \rho(\mathcal{A}z,\tau)$$

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