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Measure-expansive diffeomorphisms

Kazuhiro Sakai [a,*], Naoya Sumi [b], Kenichiro Yamamoto [c]

[a] *Department of Mathematics, Utsunomiya University, 350 Mine-machi, Utsunomiya-shi, Tochigi 321-8505, Japan*
[b] *Department of Mathematics, Kumamoto University, 2-39-1 Kurokami Chuo-ku, Kumamoto City 860-8555, Japan*
[c] *School of Information Environment, Tokyo Denki University, 2-1200 Buseigakuendai, Inzai-shi, Chiba 270-1382, Japan*

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ABSTRACT

In this paper, measure-expansive diffeomorphisms are considered and the characterizations of the C^1-interiors of the set of measure-expansive diffeomorphisms are obtained.

1. Introduction

Let (X, d) be a compact metric space, and let $f : X \to X$ be a homeomorphism. We say that f is expansive if there is a constant $c > 0$ such that $d(f^n(x), f^n(y)) \leqslant c$ $(x, y \in X)$ for all $n \in \mathbb{Z}$ implies $x = y$. The notion of expansivity has been intensively studied by several researchers, mainly from the topological view point, and lots of important fruitful results were obtained. Nowadays, expansive theory of dynamical systems has been well developed. This notion very often appears in the investigation of qualitative theory of dynamical systems, for instance, the stability theory of dynamical systems, and is usually playing an essential role in the investigation (for instance, [1,6,10] among others).

The notion of expansivity also plays an important role in the study of ergodic theory of dynamical systems, for example, the proof of the existence of Markov partitions. Very recently, the notion of the expansive measures was introduced by [7] as a generalization of the notion of expansivity.

* Corresponding author.
E-mail addresses: kazsakai@cc.utsunomiya-u.ac.jp (K. Sakai), sumi@sci.kumamoto-u.ac.jp (N. Sumi), yamamoto.k.ak@sie.dendai.ac.jp (K. Yamamoto).

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Let μ be a Borel probability measure which is not necessarily f-invariant. We say that f is *measure-expansive* (or simply, *μ-expansive*) if there is $\delta > 0$ such that for any $x \in X$, $\mu(\Gamma_\delta(x)) = 0$. Here $\Gamma_\delta(x) = \{y \in X\colon\ d(f^n(x), f^n(y)) \leqslant \delta$ for $n \in \mathbb{Z}\}$. Note that if a measure μ is non-atomic, then every expansive homeomorphism f is measure-expansive by definition.

Let g be a Denjoy map on the unit circle S^1; that is, g is a non-transitive diffeomorphism of S^1 with irrational rotation number. It is well-known that g is uniquely ergodic and the support of the measure μ is a Cantor set. It is shown by [7] that every Denjoy map g of S^1 is μ-expansive. The basic property of measure-expansive homeomorphisms was also studied therein, and different proofs of the famous results for expansive homeomorphisms were given from a view point of measure theory, for instance, the non-existence of expansive homeomorphisms on the circle (and thus, the above map g is not expansive).

In this paper, we begin to study the geometric theory of expansive differentiable dynamical systems from the measure theoretical view point in the context of hyperbolic dynamical systems. To be more precise, we characterize the dynamics of measure-expansive systems in the framework of hyperbolic dynamical systems (of course, the characterization changes depending on the measures which we consider; just probability measures or invariant ones).

Let $\mathrm{Diff}(M)$ be the space of diffeomorphisms of a closed C^∞ manifold M endowed with the C^1-topology. Denote by d the distance on M induced from the Riemannian metric $\|\cdot\|$ on the tangent bundle TM. Hereafter, let $P(f)$ be the set of periodic points of $f \in \mathrm{Diff}(M)$, and let $\Omega(f)$ be the set of non-wandering points of it.

Let $\Lambda \subset M$ be a closed f-invariant set, that is, $f(\Lambda) = \Lambda$. Recall that Λ is *hyperbolic* if the tangent bundle $T_\Lambda M$ has a Df-invariant splitting $E^s \oplus E^u$ with constants $C > 0$ and $0 < \lambda < 1$ such that

$$\left\|Df^n\big|_{E^s_x}\right\| \leqslant C\lambda^n \quad \text{and} \quad \left\|Df^{-n}\big|_{E^u_x}\right\| \leqslant C\lambda^n$$

for all $x \in \Lambda$ and $n \geqslant 0$. It is well-known that if Λ is hyperbolic, then $f_{|\Lambda}$ is expansive (see [10]).

We say that f satisfies *Axiom A* if $\Omega(f)$ is hyperbolic and $\Omega(f) = \mathrm{cl}(P(f))$, where $\mathrm{cl}(A)$ denotes the closure of a set $A \subset M$. The *stable manifold* of a point x is the set

$$W^s(x) = \left\{y \in M\colon\ d\left(f^n(x), f^n(y)\right) \to 0 \text{ as } n \to \infty\right\}.$$

The *unstable manifold* $W^u(x)$ of x is also defined analogously for $n \to -\infty$. It is well-known that if Λ is hyperbolic, then $W^s(x)$ and $W^u(x)$ are both immersed manifolds for each $x \in \Lambda$ (for instance, [10, Corollary 9.2]). We say that f is *quasi-Anosov* if for all $v \in TM \setminus \{0\}$, the set $\{\|Df^n(v)\|\colon\ n \in \mathbb{Z}\}$ is unbounded (see [5]).

Denote by $\mathcal{E}$ the set of all expansive diffeomorphisms of M. In [4], the geometric properties of expansive diffeomorphisms were studied by Mañé and he gave the following characterization on $\mathcal{E}$ (see also [8]). This result plays a basic role in this paper.

Theorem 1.1. *The following conditions are mutually equivalent*:

- *f is quasi-Anosov,*
- *f satisfies Axiom A and all intersections of stable and unstable manifolds are quasi-transversal, that is,*

$$T_xW^s(x) \cap T_xW^u(x) = \{0\} \quad \textit{for all } x \in M,$$

- *f is in the C^1-interior,* $\mathrm{int}(\mathcal{E})$, *of $\mathcal{E}$.*