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On large deviations for some sequences of weighted means of Gaussian processes ☆

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A B S T R A C T

In this paper we study some sequences of weighted means of continuous real valued Gaussian processes. More precisely we consider suitable generalizations of both arithmetic and logarithmic means of a Gaussian process with covariance function which satisfies either an exponential decay condition or a power decay condition. Our aim is to provide limits of variances of functionals of such weighted means which allow the application of some large deviation results in the literature.

1. Introduction

In this paper we consider some sequences of weighted means of continuous real valued Gaussian processes $\{Z_n\colon n \geqslant 1\}$ defined in terms of a sequence of positive coefficients $\{a_n\colon n \geqslant 1\}$ and of the random variables of a Gaussian process $X = \{X_t\colon t \geqslant 1\}$ (see Hypothesis 2.1). The covariance function of X satisfies either the exponential decay condition (**exp**) or the power decay condition (**power**) presented below. We consider suitable generalizations of both arithmetic and logarithmic means (see Hypotheses 3.1 and 3.2 under condition (**exp**), and Hypothesis 3.3 under condition (**power**)).

Our aim is to provide limits of variances of functionals of these weighted means which allow the application of some large deviation results in the literature. The theory of large deviations is a collection of techniques for the asymptotic computation of small probabilities on an exponential scale (see e.g. [6] as a reference on this topic).

In the setting of the present paper logarithmic means appear only as a particular case (see Hypotheses 3.2 and 3.3); nevertheless they can be considered as one of the motivations of our work, since they provide a strict analogy with the sequences of random measures considered in the *almost sure central limit theorem*:

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if $\{U_n\colon\ n \geqslant 1\}$ is a sequence of i.i.d. centered random variables with unit variance and if we set $X_n := \frac{U_1+\cdots+U_n}{\sqrt{n}}$, then we have the almost sure weak convergence to the standard Normal distribution of the sequences of random measures

$$\left\{\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}1_{\{X_k\in\cdot\}}\colon\ n\geqslant 1\right\} \tag{1}$$

and, of course, of

$$\left\{\frac{1}{L(n)}\sum_{k=1}^{n}\frac{1}{k}1_{\{X_k\in\cdot\}}\colon\ n\geqslant 1\right\},\quad \text{where } L(n):=\sum_{k=1}^{n}\frac{1}{k}. \tag{2}$$

The almost sure central limit theorem was proved independently in [3,9] and [24] under stronger moment assumptions; successive refinements appear in [10] and [17], in which only finite variance is required.

We also recall some references in the literature on the asymptotic behavior of weighted means. General asymptotic results can be found in [1] and [14]; other results are proved in [2] (for some logarithmic means based on the fractional Brownian motion), and in [8] (for some logarithmic means based on maxima and sums of some classes of stationary Gaussian sequences). In what concerns the references for large deviation principles, we recall [21] and [13] for the sequences (1) and (2), respectively (in both cases it is assumed that all the moments are finite, and the optimality of this assumption is discussed in [20]), [23] for the so called Lévy strong arc-sine law (see [19]), [15] and [11].

We remark that, for the examples in this paper, X_t admits a weak limit as $t \to \infty$ and, when we deal with the logarithmic means, the two different decay conditions (**exp**) and (**power**) for the covariance of X lead to the speed functions $v_n = (\log n)^2$ and $v_n = \log n$, respectively. This has some analogy with the examples presented in [11] where logarithmic means of sequences having a weak limit are considered, though in that case the speed function is always $v_n = \log n$.

Throughout this paper we use the notation $a_n \sim b_n$ to mean that $\lim_{n\to\infty}\frac{a_n}{b_n} = 1$. Moreover we use the symbol δ_t for the unit measure concentrated at a point t, and the symbol $N[\mu, \sigma^2]$ for the Normal distribution with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 \geqslant 0$ (the case $\sigma = 0$ for the case of a constant random variable μ is allowed). We also adopt the usual convention $\sum_{i=a}^{b} = 0$ for any sum such that $a > b$.

Here is the outline of the paper. We start with some preliminaries in Section 2. The results are proved in Section 3, where we also present some relevant examples related to Ornstein–Uhlenbeck and autoregressive processes, and fractional and integrated Brownian motions. Finally Section 4 is devoted to some concluding remarks.

2. Preliminaries

In this section we refer to the concept of *large deviation principle* (LDP from now on) and some other basic preliminaries on large deviations (see [6, pp. 4–5]); we also refer to the well-known Gärtner–Ellis Theorem (see e.g. Theorem 2.3.6 in [6]) and to its infinite dimensional version on a topological vector space (i.e. the Baldi Theorem; see e.g. Theorem 4.5.20 in [6]). We start with a version of the latter result where the topological vector space consists of the family $C[0,1]$ of all real valued continuous functions defined on $[0,1]$ (for simplicity we consider the time interval $[0,1]$, but one could take any other finite time interval $[a,b] \subset \mathbb{R}$). In view of this we have to consider $\mathcal{M}[0,1]$, i.e. the topological dual space of $C[0,1]$ formed by the family of all signed measures with bounded variation on $[0,1]$. Moreover we set $\langle\theta, f\rangle := \int_0^1 f(t)\,d\theta(t)$ for $f \in C[0,1]$ and $\theta \in \mathcal{M}[0,1]$, we denote by $|\theta|$ the total variation of a measure θ, and we set $\|\theta\| = |\theta|([0,1])$.

Proposition 2.1. *Let $\{Z_n\colon\ n \geqslant 1\}$ be a sequence of processes in $C[0,1]$. Assume that, for some speed function v_n (i.e. a sequence such that $\lim_{n\to\infty} v_n = \infty$), the limit*