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On a functional equation connected to Hermite quadrature rule

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ABSTRACT

In this paper we deal with the functional equation

$$F(y)-F(x)=(y-x)\left[\alpha f(x)+\beta f\left(\frac{x+y}{2}\right)+\alpha f(y)\right]+(y-x)^2\left[g(y)-g(x)\right],$$

which is connected to Hermite quadrature rule. It is easy to note that particular cases of this equation generalize many well known functional equations connected to quadrature rules and mean value theorems. Thus the set of solutions is too complicated to be described completely and therefore we prove that (under some assumptions) all solutions of the above equation must be polynomials. We obtain the aforementioned result using a lemma proved by M. Sablik, however this lemma works only in case $\beta \neq 0$. Taking $\beta = 0$, we obtain the following equation

$$F(y)-F(x)=(y-x)\left[f(x)+f(y)\right]+(y-x)^2\left[g(y)-g(x)\right],$$

which is also solved in the paper.

1. Introduction

In recent papers (see for example [2,6–8,10,11]) functional equations of the form

$$F(y)-F(x)=(y-x)\sum_{i=1}^{n}a_i f(\alpha_i x+\beta_i y),\quad x,y\in\mathbb{R},$$

were studied in many particular cases. Equations of this type are connected to many well known quadrature rules (such as quadrature rule of Simpson, Gauss, Lobatto, Radau and others).

In the current paper we make a step forward considering functional equation generated by the Hermite quadrature formula. In this quadrature rule we use not only values of the considered function but also values of its derivative. Thus we approximate the definite integral of f in the following manner

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$$\int_x^y f(t)\,dt \approx \frac{y-x}{n}\left[\frac{1}{2}\big(f(x)+f(y)\big)+f(x_1)+\cdots+f(x_{n-1})\right]$$
$$+\frac{(y-x)^2}{12}\big[f'(x)-f'(y)\big], \tag{1}$$

where $x_1,\ldots,x_{n-1}$ are distinct points lying in the interval (x,y).

Taking here $n=1$, we get the so-called *corrected trapezoidal formula*

$$\int_x^y f(t)\,dt \approx \frac{y-x}{2}\big[f(x)+f(y)\big]+\frac{(y-x)^2}{12}\big[f'(x)-f'(y)\big]. \tag{2}$$

Similarly as in previous papers we get from (2) functional equation

$$F(y)-F(x)=(y-x)\big[f(x)+f(y)\big]+(y-x)^2\big[g(y)-g(x)\big], \tag{3}$$

which will be solved later in the paper. Next we consider equation connected to (1) with $n=2$ and $x_1=\frac{x+y}{2}$

$$F(y)-F(x)=(y-x)\left[\alpha f(x)+\beta f\left(\frac{x+y}{2}\right)+\alpha f(y)\right]+(y-x)^2\big[g(y)-g(x)\big]. \tag{4}$$

It is worth noticing that this equation generalizes many well known equations such as:

$$F(y)-F(x)=(y-x)f\left(\frac{x+y}{2}\right)$$

considered in [1],

$$F(y)-F(x)=(y-x)\big[f(x)+f(y)\big]$$

see [4],

$$F(y)-F(x)=(y-x)\left[\frac{1}{6}f(x)+\frac{2}{3}f\left(\frac{x+y}{2}\right)+\frac{1}{6}f(y)\right] \tag{5}$$

see for example [3,5,12] and many others. Therefore the set of solutions of (4) will be extremely complicated and, consequently, it seems not possible to provide a full description of solutions of this equation. Thus we only prove that solutions of this equation under some assumptions on α, β must be polynomials. If we know that all solutions are polynomials then it is easy to find the concrete forms of these solutions in particular cases of (4).

2. Preliminaries

At the beginning we will cite a lemma from the paper of A. Lisak and M. Sablik [10]. First we introduce some notations. Let G and H be commutative groups, then $SA^i(G;H)$ denotes the group of all i-additive, symmetric mappings from G^i into H for $i \geqslant 2$, while $SA^0(G;H)$ denotes the family of constant functions from G to H and $SA^1(G;H)=\mathrm{Hom}(G;H)$. We also denote by $\mathcal{I}$ the subset of $\mathrm{Hom}(G;G)\times\mathrm{Hom}(G;G)$ containing all pairs $(\alpha;\beta)$ for which $\mathrm{Ran}(\alpha)\subset\mathrm{Ran}(\beta)$. Furthermore, we adopt a convention that a sum over empty set of indices equals 0. The following lemma proved in [10] is a generalization of lemma presented in [13].