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Hypergroups and invariant complemented subspaces

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ABSTRACT

Let K be a hypergroup with a Haar measure. The purpose of the present paper is to initiate a systematic approach to the study of the class of invariant complemented subspaces of $L_\infty(K)$ and $C_0(K)$, the class of left translation invariant w^*-subalgebras of $L_\infty(K)$ and finally the class of non-zero left translation invariant C^*-subalgebras of $C_0(K)$ in the hypergroup context with the goal of finding some relations between these function spaces. Among other results, we construct two correspondences: one, between closed Weil subhypergroups and certain left translation invariant w^*-subalgebras of $L_\infty(K)$, and another, between compact subhypergroups and a specific subclass of the class of left translation invariant C^*-subalgebras of $C_0(K)$. By the help of these two characterizations, we extract some results about invariant complemented subspaces of $L_\infty(K)$ and $C_0(K)$.
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1. Introduction

A hypergroup is a locally compact Hausdorff space equipped with a convolution product which maps any two points to a probability measure with a compact support. Hypergroups generalize locally compact groups in which the above convolution reduces to a point mass measure. It was in the 1970s that Dunkl [4], Jewett [10] and Spector [23] began the study of hypergroups (in [10] they are called convos). The theory of hypergroups then developed in various directions, namely in the area of commutative hypergroups [5,18], specifically orthogonal polynomials [11,29], function spaces [28] and weighted hypergroups [7]. It is worthwhile to mention that there are some axiomatic differences in the definition of hypergroups given by these three authors, however, the core idea remains the same. Since almost all of the analysis on hypergroups have been based on the definition of Jewett, we shall base our work on his definition. For a complete history, we refer the interested reader to [19]. Throughout, K will denote a hypergroup with a left Haar measure λ.

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The class of F-algebras, also known as Lau algebras was created and analyzed in 1983 [15]. This rich structure contains the Fourier algebra $A(G)$, the Fourier–Stieltjes algebra $B(G)$, the group algebra $L_1(G)$ and the measure algebra $M(G)$ of a locally compact group G. It also contains the hypergroup algebra $L_1(K)$ and the measure algebra $M(K)$. Lau introduced left amenable F-algebras [15, §4] and provided various characterizations of this object (see also [17]). By [22, Theorem 3.2] K is amenable if and only if $L_1(K)$ is left amenable. On the other hand, $L_1(K)$ demonstrate different behaviors in terms of amenability and weak amenability [22,12] in comparison with its group counterpart. Amenability of hypergroups was also studied in [27]. Let $L_\infty(K)$ be the w^*-algebra of all essentially bounded measurable complex-valued functions on K with essential supremum norm and point-wise multiplication and let Y be a closed, translation invariant subspace of $L_\infty(K)$. A closed, left translation invariant subspace X of Y is said to be invariantly complemented in Y if X is the range of a continuous projection on Y, which commutes with all left translation operators on Y. This concept was introduced by Lau [14] for locally compact groups and was studied in [6]. Motivated by the harmonic analysis considered by Lau [14] and Lau and Losert [16], we initiate the study of the class of invariant complemented subspaces of $L_\infty(K)$ and $C_0(K)$, the class of non-zero left translation invariant C^*-subalgebras of $C_0(K)$ and finally the class of left translation invariant w^*-subalgebras of $L_\infty(K)$ in the hypergroup context with the goal of finding some relations between these function spaces.

Let X be a left translation invariant w^*-subalgebra of $L_\infty(K)$. Takesaki and Tatsuuma in 1971 showed that $X = \{f \in L_\infty(K) \mid R_n f = f,\ \forall n \in N\}$, for a unique closed subgroup N of a locally compact group K ([24, Theorem 2], see also [14, Lemma 3.2] for a different proof). A decade later, Lau [14, Theorem 3.3] proved that K is amenable if and only if X is invariantly complemented in $L_\infty(K)$, where K is a locally compact group. In Section 3 we shall initiate a formal study of the class of left translation invariant w^*-subalgebras of $L_\infty(K)$ in the hypergroup setting. In the process of building a bridge between this class and closed subhypergroups, by the nature of our framework, we encounter a feature which is dormant in the group context. We note that the constructed w^*-subalgebra X has a certain property that we assume for obtaining a reasonable correspondence. This new notion, "local translation property TB", extends the notion of translation property TB, which was considered by Voit [26]. In the main theorem of this section, we prove that X is a left translation invariant w^*-subalgebra of $L_\infty(K)$ such that $X \cap CB(K)$ has the local translation property TB if and only if there exists a unique closed Weil subhypergroup N such that $X = \{f \in L_\infty(K) \mid R_g f = R_k f,\ \forall g \in k * N,\ k \in K\}$, where R_g is the right translation operator for $g \in K$. Furthermore, the normality of N is characterized by X being translation invariant and inversion invariant (Theorem 3.11). As a consequence, then we show that every left translation invariant w^*-subalgebra of $L_\infty(K)$ such that $X \cap CB(K)$ has the local translation property TB is invariantly complemented in $L_\infty(K)$, where K is a compact hypergroup (Corollary 3.13).

Let X be a non-zero left translation invariant C^*-subalgebra of $C_0(K)$. DeLeeuw [3, Theorem 5.1] proved that X is the algebra $C_0(K/N)$, for some subgroup N of K, if K is a commutative locally compact group. Lau and Losert [16, Lemma 12] extended their result to any locally compact group K. In Section 4, we commence an investigation of the class of non-zero left translation invariant C^*-subalgebras of $C_0(K)$, when K is furnished with a hypergroup structure. We first set up a basis by providing a characterization of hypergroups admitting an invariant mean on the space $WAP(K)$, the space of all continuous weakly almost periodic functions on K (Lemma 4.2). As another fundamental result, we endow X with the local translation property TB and we prove that X is a non-zero left translation invariant C^*-subalgebra of $C_0(K)$ with the local translation property TB if and only if there exists a unique compact subhypergroup N of K such that $X = \{f \in C_0(K) \mid R_n f = f,\ \forall n \in N\}$ (Lemma 4.6). Then in one of our major results we show in particular that X is invariantly complemented in $C_0(K)$ provided that X has the local translation property TB (Theorem 4.8).

Finally, in Section 5 we provide a one-to-one correspondence between non-zero weak*-closed, $C_0(K)$-invariant $*$-subalgebras of $M(K)$ and closed subhypergroups (Remark 5.1), left translation invariant strictly

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