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Backward–backward splitting in Hadamard spaces

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ABSTRACT

The backward–backward algorithm is a tool for finding minima of a regularization of the sum of two convex functions in Hilbert spaces. We generalize this setting to Hadamard spaces and prove the convergence of an error-tolerant version of the backward–backward method.

1. Introduction

Proximal splitting methods provide powerful techniques for solving non-differentiable convex optimization problems in Hilbert spaces, see e.g. [12] for a survey on this topic in the context of signal processing.

Recently, Bačák et al. [5,3] investigated the convergence of the proximal point algorithm and the alternating projection method for convex functions in Hadamard spaces, which are also known as complete CAT(0) spaces. The aim of this work is to generalize both approaches and to give an example of a proximal splitting method in an Hadamard space, see [4] for another one in locally compact Hadamard spaces.

This work is organized as follows: In Section 2, we set up our terminology. Particular attention is given to the geometry of Hadamard spaces (Section 2.1), where we mention inequality (3), which is a stronger form of the triangle inequality in CAT(0) spaces. It enables us to generalize some well-known facts from Hilbert to Hadamard spaces, though its proof is elementary. In Section 2.2, the emphasis rests on convex functions, where our terminology is adopted from e.g. [6] in the context of Hilbert spaces and might be unusual in the community of CAT(0) spaces. Section 2.3 is devoted to weak convergence. For the history of generalizing weak convergence from Hilbert to Hadamard spaces, see e.g. [5, Section 2.3]. In Section 3, we present a convergence analysis of the backward–backward algorithm and show its tolerance with respect to summable error sequences.

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2. Preliminaries

2.1. Geometry of an Hadamard space

An *Hadamard space* (X, d) is a complete metric space, where to each two points $x, y \in X$ a *midpoint* $m \in X$ can be assigned such that

$$d(z,m)^2 \leqslant \frac{1}{2}d(z,x)^2 + \frac{1}{2}d(z,y)^2 - \frac{1}{4}d(x,y)^2 \tag{1}$$

for all $z \in X$. If X is a closed, convex subset of a Hilbert space with the metric induced by the inner product, relation (1) holds with equality for $m = \frac{1}{2}(x+y)$ and all $z \in X$ by the parallelogram identity.

More generally, for each two points $x, y \in X$, there is a map $\gamma_{x,y} : [0,1] \to X$, such that

$$d\big(z, \gamma_{x,y}(\lambda)\big)^2 \leqslant (1-\lambda)d(z,x)^2 + \lambda d(z,y)^2 - \lambda(1-\lambda)d(x,y)^2 \tag{2}$$

for all $z \in X$ and $\lambda \in [0,1]$. The curve $\gamma_{x,y}$ is uniquely determined and called the *geodesic* joining x and y. It holds

$$d\big(\gamma_{x,y}(\lambda_1), \gamma_{x,y}(\lambda_2)\big) = |\lambda_1 - \lambda_2| d(x,y)$$

for $0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant 1$, $\gamma_{x,y}(0) = x$ and $\gamma_{x,y}(1) = y$. The *geodesic segment* joining $x \in X$ and $y \in X$ is defined as

$$[x,y] := \big\{\gamma_{x,y}(\lambda) \bigm| 0 \leqslant \lambda \leqslant 1\big\}.$$

From (2) (or (1)) one obtains a useful inequality by

$$\begin{aligned} 0 &\leqslant d\left(\gamma_{x,y}\left(\frac{1}{2}\right), \gamma_{z,w}\left(\frac{1}{2}\right)\right)^2 \\ &\leqslant \frac{1}{2}d\left(\gamma_{x,y}\left(\frac{1}{2}\right), z\right)^2 + \frac{1}{2}d\left(\gamma_{x,y}\left(\frac{1}{2}\right), w\right)^2 - \frac{1}{4}d(z,w)^2 \\ &\leqslant \frac{1}{4}\big(d(x,z)^2 + d(y,z)^2 + d(x,w)^2 + d(y,w)^2 - d(x,y)^2 - d(z,w)^2\big) \end{aligned}$$

for all $x, y, z, w \in X$, which yields

$$d(x,y)^2 + d(z,w)^2 \leqslant d(x,z)^2 + d(x,w)^2 + d(y,z)^2 + d(y,w)^2. \tag{3}$$

Berg and Nikolaev [9] proved that, for metric spaces, the inequalities (1) and (3) are in fact equivalent.

For two Hadamard spaces X and Y, the Cartesian product $X \times Y$ is an Hadamard space with the metric given by

$$d\big((x_1,y_1),(x_2,y_2)\big)^2 = d(x_1,x_2)^2 + d(y_1,y_2)^2$$

and the geodesics

$$\gamma_{(x_1,y_1),(x_2,y_2)}(\lambda) = \big(\gamma_{x_1,x_2}(\lambda), \gamma_{y_1,y_2}(\lambda)\big)$$

for $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. We shall write X^2 for $X \times X$.

In what follows, let X be an Hadamard space.