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On a backward parabolic problem with local Lipschitz source

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ABSTRACT

We consider the regularization of the backward in time problem for a nonlinear parabolic equation in the form $u_t + Au(t) = f(u(t), t)$, $u(1) = \varphi$, where A is a positive self-adjoint unbounded operator and f is a local Lipschitz function. As known, it is ill-posed and occurs in applied mathematics, e.g. in neurophysiological modeling of large nerve cell systems with action potential f in mathematical biology. A new version of quasi-reversibility method is described. We show that the regularized problem (with a regularization parameter $\beta > 0$) is well-posed and that its solution $U_\beta(t)$ converges on $[0, 1]$ to the exact solution $u(t)$ as $\beta \to 0^+$. These results extend some earlier works on the nonlinear backward problem.

1. Introduction

Let H be a Hilbert space with the inner product $\langle .,. \rangle$ and the norm $\|.\|$. In this paper, we consider the backward nonlinear parabolic problem of finding a function $u : [0, 1] \to H$ such that

$$\begin{cases} u_t + Au = f\big(u(t), t\big), & 0 < t < 1, \\ u(1) = \varphi, \end{cases} \tag{1}$$

where the function f is defined later and the operator A is self-adjoint on a dense space $D(A)$ of H such that $-A$ generates a compact contraction semi-group on H. The backward parabolic problems arise in different forms in heat conduction [4,10], material science [16], hydrology [3] and also in many other practical applications of mathematics and engineering sciences. If $H = L^2(0, l)$ for $l > 0$, $A = -\Delta$ and $f(u(t), t) = u\|u\|^2_{L^2(0,l)}$ then a concrete version of problem (1) is given as

$$\begin{cases} u_t - \Delta u = u\|u\|^2_{L^2(0,l)}, & (x, t) \in (0, l) \times (0, 1), \\ u(0, t) = u(l, t) = 0, & t \in (0, 1), \\ u(x, 1) = \varphi(x), & x \in (0, l). \end{cases} \tag{2}$$

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The first equality in problem (2) is a semilinear heat equation with cubic-type nonlinearity and has many applications in computational neurosciences. It occurs in neurophysiological modeling of large nerve cell systems in mathematical biology (see [17]).

Let $u(t)$ be the (unknown) solution of (1), continuous on $t \geqslant 0$ to H with an (unknown) initial value $u(0)$. In practice, $u(1)$ is known only approximately by $\varphi \in H$ with $\|u(1) - \varphi\| \leqslant \beta$, where the constant β is a known small positive number. This problem is well known to be severely ill-posed [15] and regularization methods are required. The homogeneous linear case of problem (1)

$$\begin{cases} u_t + Au = 0, & 0 < t < 1, \\ u(1) = \varphi, \end{cases} \tag{3}$$

has been considered in many papers, such as [2,1,6–9,11–13,18] and references therein. For nonlinear case, there are not many results devoted to backward parabolic equations. In [20,21], under assumptions that $f : H \times \mathbb{R} \to H$ is a global Lipschitz function with respect to the first variable u, i.e. there exists a positive number $k > 0$ independent of $w, v \in H$, $t \in \mathbb{R}$ such that

$$\big\|f(w,t) - f(v,t)\big\| \leqslant k\|w - v\|, \tag{4}$$

we regularized problem (1) and gave some error estimates. To improve the convergence of our method, P.T. Nam [14] gave another method to get the Hölder estimate for regularized solution. More recently, Hetrick and Hughes [5] established some continuous dependence results for nonlinear problem. Their results are also solved under the assumption (4). Until now, to our knowledge, we did not find any papers dealing with the backward parabolic equations included the local Lipschitz source f.

In this paper, we propose a new modified quasi-reversibility method to regularize (1) in case of the local Lipschitz function f. The techniques and methods in previous papers on global Lipschitz function cannot be applied directly to solve the problem (1). The main idea of the paper is of replacing the operator A in (1) by an approximated operator A_β, which will be defined later. Then, using some new techniques, we establish the following approximation problem

$$\begin{cases} v_\beta'(t) + A_\beta v_\beta(t) = f\big(v_\beta(t), t\big), & 0 < t < 1, \\ v_\beta(1) = \varphi, \end{cases} \tag{5}$$

and give an error estimate between the regularized solution of (5) and the exact solution of (1).

Namely, assume that A admits an orthonormal eigenbasis $\{\phi_k\}$ on H corresponding to the eigenvalues $\{\lambda_k\}$ of A; i.e. $A\phi_k = \lambda_k\phi_k$. Without loss of generality, we shall assume that

$$0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots, \quad \lim_{k\to\infty} \lambda_k = \infty.$$

For every v in H having the expansion $v = \sum_{k=1}^{\infty} \langle v, \phi_k\rangle\phi_k$, we define

$$A_\beta(v) = \sum_{k=1}^{\infty} \ln^+\left(\frac{1}{\beta\lambda_k + e^{-\lambda_k}}\right)\langle v, \phi_k\rangle\phi_k,$$

where $\ln^+(x) = \max\{\ln x, 0\}$. And for $0 \leqslant t \leqslant s \leqslant T$, we define

$$G_\beta(t,s)(v) = \sum_{k=1}^{\infty} \max\big\{\big(\beta\lambda_k + e^{-\lambda_k}\big)^{t-s}, 1\big\}\langle v, \phi_k\rangle\phi_k.$$