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Mixed variational formulations in locally convex spaces

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ABSTRACT

The main purpose of this paper is to extend to the setting of locally convex spaces the study of the mixed variational formulation of some elliptic boundary value problems, the so-called *Babuška–Brezzi theory*. This study consists of characterizing the existence of a solution and giving conditions that guarantee the stability of the corresponding Galerkin method.

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1. Introduction

The classical Lax–Milgram theorem provides a sufficient condition – coercivity – for a continuous and bilinear form on a Hilbert space, in order that it represents the whole space. However this central result of variational analysis, as well as its extensions to continuous bilinear forms a defined on the product of more general Banach spaces $E \times F$ (see for instance [14, Theorem 12]), is restrictive, since it only analyzes the dichotomy: either

$$\text{each } y_0^* \in F^* \text{ is of the form } a(x_0, \cdot) \text{ for some } x_0 \in E$$

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or

$$\text{there exists some } y_0^* \in F^* \text{ such that for all } x_0 \in E,\ a(x_0, \cdot) \neq y_0^*.$$

However, recently ([19, Theorem 1.2], see also [21, Theorem 2.2]) the Lax–Milgram theorem has been extended in two different ways, both generalizing the kind of space, the locally convex ones, and fixing one continuous linear functional. This way, it is concerned with the natural alternative: given a *specific* $y_0^* \in F^*$, either

$$\text{there exists } x_0 \in E \text{ such that } a(x_0, \cdot) = y_0^*$$

or

$$\text{for all } x_0 \in E, \quad a(x_0, \cdot) \neq y_0^*.$$

Such a generalization of Lax–Milgram's theorem [19, Theorem 1.2] characterizes when one of these possibilities occurs, in terms of the existence of a constant in the normed framework.

A similar and more general situation arises when dealing with mixed variational formulations of some elliptic boundary value problems, thus leading to consider the following question: given Hilbert spaces E and F (over the scalar field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), continuous linear functionals $x_0^* : E \longrightarrow \mathbb{K}$ and $y_0^* : F \longrightarrow \mathbb{K}$ and continuous bilinear forms $a : E \times E \longrightarrow \mathbb{K}$ and $b : E \times F \longrightarrow \mathbb{K}$, find if possible $(x_0, y_0) \in E \times F$ with

$$\begin{cases} x \in E \quad \Rightarrow \quad x_0^*(x) = a(x_0, x) + b(x, y_0), \\ y \in F \quad \Rightarrow \quad y_0^*(y) = b(x_0, y). \end{cases}$$

A detailed bibliography describes conditions under which this mixed variational problem admits a unique solution, as well as the analysis of its Galerkin schemes (see for instance [2,7,8,18,5]): it is the so-called *Babuška–Brezzi theory*. In this work we adopt a more abstract approach for the mixed variational problem, in the context of locally convex spaces. In fact, we study a more general type of mixed variational problem that, in the particular setting of Hilbert spaces, relies on the following construction: let E, F, G and H be Hilbert spaces, let $y_0^* : F \longrightarrow \mathbb{K}$ and $w_0^* : H \longrightarrow \mathbb{K}$ be continuous linear functionals and let $a : E \times F \longrightarrow \mathbb{K}$, $b : F \times G \longrightarrow \mathbb{K}$ and $c : E \times H \longrightarrow \mathbb{K}$ be continuous bilinear forms. Is it possible to find a solution $(x_0, z_0) \in E \times G$ of

$$\begin{cases} y \in F \quad \Rightarrow \quad y_0^*(y) = a(x_0, y) + b(y, z_0), \\ w \in H \quad \Rightarrow \quad w_0^*(w) = c(x_0, w)? \end{cases}$$

Specifically, Section 2 deals with characterizing when this mixed variational problem, for fixed $y_0^* : F \longrightarrow \mathbb{R}$ and $w_0^* : H \longrightarrow \mathbb{R}$, admits a solution in terms of the existence of a certain continuous seminorm, a characterization which when concreted for normed spaces, entails the existence of an adequate scalar. In addition, an explicit control of the norm of a solution – the solution, if it is unique – is derived. In Section 3 we establish this type of result, besides its global version, in the sense that we obtain necessary and sufficient conditions, not only for some fixed y_0^* and w_0^*, but also for all continuous and linear functionals on F and H, in order that the mixed variational problem admits a solution. As a particular case, we return to some of the classical results of the Hilbert framework. Moreover, in Example 3.8 we make use of our results in order to prove the existence of one and only one solution of a mixed variational formulation of an elliptic boundary value problem, for which the classical Hilbertian theory of Babuška and Brezzi does not apply. Continuing in the normed case, in Section 4 we develop the corresponding Galerkin scheme, arriving at some stability results and providing a numerical testing based upon the use of bases in appropriate Banach spaces.