J. Math. Anal. Appl. 414 (2014) 850–870

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

An alternative approach to solve the mixed AKNS equations

Dianlou Du, Xiao Yang *

School of Mathematics and Statistics, Zhengzhou University, Zhengzhou, Henan 450001, PR China

A R T I C L E I N F O

Article history:
Received 15 February 2013
Available online 21 January 2014
Submitted by G. Bluman

Keywords:
Lenard operator
The mixed AKNS equation
Integrable system
Algebraic–geometric solution

A B S T R A C T

The algebraic–geometric solutions of the mixed AKNS equations are investigated through a finite-dimensional Lie–Poisson Hamiltonian system, which is generated by the nonlinearization of the adjoint equation related to the AKNS spectral problem. First, each mixed AKNS equation can be decomposed into two compatible Lie–Poisson Hamiltonian flows. Then the separated variables on the coadjoint orbit are introduced to study these Lie–Poisson Hamiltonian systems. Further, based on the Hamilton–Jacobi theory, the relationship between the action-angle coordinates and the Jacobi-inversion problem is established. In the end, using Riemann–Jacobi inversion, the algebraic–geometric solutions of the first three mixed AKNS equations are obtained.

1. Introduction

The Lenard recursion formula is an effective approach to the integrable systems. It was first used to give the KdV hierarchy in the work of Gardner, Greene, Kruskal and Miura [10]. Then based on the recursion relation of Lenard, a new proof for the existence of an infinite sequence of conserved functionals for the KdV equation was presented by Lax [19]. After that, there came other examples [16,23]. The concept of the bi-Hamiltonian theory appeared in the paper by Magri [20] and followed the paper by Gelfand and Dorfman [11], and in [11], the interrelation between the Lenard operator and the bi-Hamiltonian theory was given. Apart from the above applications to the theory of infinite dimensional integrable systems, the Lenard scheme is also widely used in the related field of mathematical physics, for example, the multi-Hamiltonian theory, the theory of Lie bialgebras, the finite dimensional integrable Hamiltonian systems and so on [1–4, 7,9,21,22,24]. A complete bibliography and a historical review on the Lenard scheme can be seen in [25].

Till now, the Lenard operators have played an important role in the study of integrability of the nonlinear evolutionary equation. The special solution series of the Lenard recursion equations are used to define the vector fields of the nonlinear evolutionary equations. Generally, the Lenard recursion equations are expressed as

$$J\xi_0 = 0, \qquad J\xi_{j+1} = K\xi_j, \quad j = 0, 1, \ldots .$$

* Corresponding author.
E-mail addresses: ddl@zzu.edu.cn (D. Du), yx@zzu.edu.cn (X. Yang).

0022-247X/$ – see front matter
http://dx.doi.org/10.1016/j.jmaa.2014.01.041

Now there is a way to yield the new series of recursion equations: exchange the Lenard operators K and J to give the new Lenard recursion equations

$$K\xi_{-1} = 0, \qquad J\xi_{-j-1} = K\xi_{-j}, \quad j = 1, 2, \ldots.$$

As a result, there are two nonlinear evolutionary equation hierarchies related to two Lenard recursion equations. Customarily, we distinguish the old nonlinear evolutionary equation and the new one by the positive flow and the negative flow respectively, and denote the two Lenard recursion equations as well. Follow this procedure, the mixed flows of the positive and the negative flows are also a meaningful topic to be concerned [15,17,18,29].

For example, in this paper, we denote the positive special solutions by $\{g_j\}$, and the vector fields by $X_j = PJg_j$, P is a projection map. Then the nonlinear evolutionary equations are written as

$$\begin{pmatrix} u \\ v \end{pmatrix}_{\tau_m} = X_{m+1}, \quad m = 1, 2, \ldots.$$

If denote the negative special solutions by $\{g_{-j}\}$, and the vector fields by $X_{-j} = PJg_{-j}$, the negative nonlinear evolutionary equations are

$$\begin{pmatrix} u \\ v \end{pmatrix}_{\tau_{-m}} = X_{-m}, \quad m = 1, 2, \ldots.$$

Further, the mixed hierarchy is given by the suitable combination of X_{m+1} and X_{-1}

$$\begin{pmatrix} u \\ v \end{pmatrix}_{\tilde{\tau}_m} = X_{m+1} - X_{-1}, \quad m = 1, 2, \ldots.$$

Along this direction, there have been many works concern on the associated object. In [26], the positive order and the negative order MKdV hierarchies are given. In [14], the self-induced transparency equations is interpreted as a generating function of the negative flow in the AKNS system, and from which arises the Heisenberg model equations. In [6] and [28], the negative flow of Kaup–Newell system is obtained, which is used to study some discrete integrable hierarchy.

In the present paper, we will study the mixed AKNS equations which are presented in [12]. In [12], the authors use the asymptotic properties and the algebraic–geometric characters of the meromorphic function, the Baker–Akhiezer vector and the hyperelliptic curve [13] to solve the equations. Now, we will employ an alternative approach: the finite dimensional Hamiltonian system theory. Firstly, correspond to the Lie–Poisson structure of the Lie algebra $sl(2,\mathbb{R})$, a Lie–Poisson Hamiltonian system related to the AKNS spectral problem is introduced and its Liouville integrability is proved. Accordingly, the mixed AKNS hierarchy is decomposed into the finite dimensional Hamiltonian flows, which are the nonlinearization of the mixed spectral problems. Secondly, the new canonical coordinates on the coadjoint orbit are given to separate these Hamiltonian systems. Further, the Hamilton–Jacobi equation for the generating function of integrals of motion is presented and the action-angle coordinates are defined. At last, the algebraic curve is introduced to give the Abel–Jacobi coordinates, in the light of Riemann–Jacobi inverse theory, the algebraic–geometric solutions of the first three mixed AKNS equations are given.

2. Notation and conventions

Let us recall the notation of Lie–Poisson structure associated with the Lie algebra $sl(2,\mathbb{R})$.

Consider a basis of Lie algebra $sl(2,\mathbb{R})$

$$E_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad E_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.1}$$