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Linear estimates for trajectories of state-constrained differential inclusions and normality conditions in optimal control

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ABSTRACT

We consider solutions to a differential inclusion $\dot{x} \in F(x)$ constrained to a compact convex set Ω. Here F is a compact, possibly non-convex valued, Lipschitz continuous multifunction, whose convex closure $\operatorname{co} F$ satisfies a strict inward pointing condition at every boundary point $x \in \partial\Omega$. Given a reference trajectory $x^*(.)$ taking values in an ε-neighborhood of Ω, we prove the existence of a second (approximating) trajectory $x : [0,T] \mapsto \Omega$ which satisfies the linear estimate $\|x(.) - x^*(.)\|_{\mathcal{AC}([0,T])} \leqslant K\varepsilon$, if one of the following two cases occurs: (i) the initial datum $x(0) = x_0$ is given, but lies in a compact set containing only points where the boundary $\partial\Omega$ is smooth; (ii) the initial point $x(0) \in \Omega$ of the approximating trajectory $x(.)$ can be chosen arbitrarily. Subsequently we employ these linear $\mathcal{AC}$-estimates to establish conditions for normality of the generalized Euler–Lagrange condition for optimal control problems with state constraints, in which we have an integral term in the cost. We finally provide an illustrative example which underlines the fact that, if conditions (i) and (ii) above are not satisfied, then we can find a degenerate minimizer.

1. Introduction

Consider a Lipschitz continuous multifunction $x \rightsquigarrow F(x) \subset \mathbb{R}^n$ with compact, possibly non convex values. A Carathéodory solution to the differential inclusion

$$\dot{x} \in F(x) \tag{1.1}$$

will be called an F-trajectory. This is an absolutely continuous map $x(.)$ from a time interval $[a,b]$ into $\mathbb{R}^n$, whose derivative $\dot{x}(t) = \frac{d}{dt}x(t)$ satisfies the differential inclusion (1.1) at a.e. time $t \in [a,b]$.

Given a closed set $\Omega \subset \mathbb{R}^n$, we wish to approximate an arbitrary F-trajectory $x^*(.)$ by a second approximating F-trajectory $x(.)$ which is *feasible*, i.e. remains inside Ω. More precisely, consider an F-trajectory $x^*(.)$ on the time interval $[0,T]$ such that

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$$d\big(x^*(t), \Omega\big) \leqslant \varepsilon \quad \text{for all } t \in [0,T]. \tag{1.2}$$

In addition, let initial data x_0 be given, satisfying

$$x_0 \in \Omega, \quad \big|x_0 - x^*(0)\big| \leqslant \varepsilon. \tag{1.3}$$

We seek a second F-trajectory $x(.) : [0,T] \mapsto \Omega$ which satisfies the initial condition

$$x(0) = x_0 \tag{1.4}$$

and remains as close as possible to $x^*(.)$ throughout the interval $[0,T]$.

It has been recently proved in [11] that, whenever the multifunction $F(.)$ is Lipschitz continuous with non-empty compact values and Ω is a compact convex set (and an inward pointing condition for the convexified of $F(.)$ is verified), then for every F-trajectory $x^*(.)$ satisfying (1.2) and every initial point x_0 as in (1.3), one can find a second F-trajectory $x(.) : [0,T] \mapsto \Omega$ such that $x(0) = x_0$ and

$$\big\|x(.) - x^*(.)\big\|_{\mathcal{AC}([0,T])} \leqslant K\varepsilon\big(1 + |\ln\varepsilon|\big), \tag{1.5}$$

where K is a constant, which does not depend on the choice of $x^*(.)$, and $\|.\|_{\mathcal{AC}([0,T])}$ is the norm in the space of absolutely continuous functions taking values in $\mathbb{R}^n$ (cf. (1.8) below). A counter-example in [3] shows that, if no additional assumption is imposed, the distance estimate (1.5) is sharp. More precisely, we can find a constant multifunction F, a (compact) convex set Ω and, for each $\varepsilon > 0$ small, an F-trajectory $x^*(.)$ which violates the state constraint Ω, satisfies (1.2), and such that, for some constant $K_0 > 0$,

$$\big\|x(.) - x^*(.)\big\|_{\mathcal{AC}([0,T])} \geqslant K_0\varepsilon|\ln\varepsilon|$$

for all F-trajectories $x(.)$ with $x(0) = x^*(0)$. (In this example the left-end point $x^*(0)$ belongs to the set in which the boundary $\partial\Omega$ is not smooth.)

The first aim of the present paper is to show that the "$\varepsilon\ln\varepsilon$" estimate (1.5) can be replaced by a linear one (w.r.t. ε), if one of the following two cases occurs:

(I) The initial point $x^*(0)$ of the reference trajectory ranges in a compact domain $\Omega^\dagger \subset \Omega_0$, where $\Omega_0 \subset \Omega$ is the set which contains the points where the boundary of Ω is smooth, and all the interior points of Ω.

(II) The constraint (1.3) is removed. More precisely, the initial condition $x(0) \in \Omega$ can be freely chosen.

Estimates like (1.5) are of great interest in Optimal Control theory, particularly in the case when they can be established in the *linear* form w.r.t. ε (observe that (1.5) is in fact a 'super-linear' estimate), for they provide insights into a number of important problems which involve state constraints (cf. [3,4,6–9,13–17,19–22]). The applications include deriving refined and unrestrictive conditions under which first order necessary conditions (the Maximum Principle or the generalized Euler–Lagrange condition) are non-degenerate or normal, characterizing value functions as (possibly unique) solutions to the Hamilton Jacobi equations, establishing 'sensitivity relations' in which the costate trajectory and the Hamiltonian are interpreted in terms of generalized gradients of the value function, and determining regularity properties of value functions.

In this paper we establish the validity of the generalized Euler–Lagrange condition in the normal form for a class of state constrained optimal control problems in which an integral term appears in the cost. It has been earlier proved that estimates having linear behaviour yield non-degeneracy of the necessary conditions for optimality (see [20], cf. [22]), or even normality (i.e. the necessary conditions apply with a