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Energy decay for the linear Klein–Gordon equation and boundary control [☆]

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ABSTRACT

In this work we study the asymptotic behavior of the solutions of the linear Klein–Gordon equation in $\mathbb{R}^N$, $N \geqslant 1$. We prove that local energy of solutions to the Cauchy problem decays polynomially. Afterwards, we use the local decay of energy to study exact boundary controllability for the linear Klein–Gordon equation in general bounded domains of $\mathbb{R}^N$, $N \geqslant 1$.

1. Introduction

Let $N \geqslant 1$ be an integer, $c \geqslant 0$ and $L = \frac{\partial^2}{\partial t^2} - \Delta + c^2$, where Δ is the Laplacian in $\mathbb{R}^N$. Given $u_0, u_1 \in C_0^\infty(\mathbb{R}^N)$ let u be the classical solution of the Cauchy problem

$$Lu(x,t) = 0, \qquad u(x,0) = u_0(x), \qquad \frac{\partial u}{\partial t}(x,0) = u_1(x), \quad x \in \mathbb{R}^N,\ t \in \mathbb{R}. \tag{1}$$

The energy of the solution u, confined in a bounded domain Ω of $\mathbb{R}^N$, at time t is given by

$$E(t) = \frac{1}{2} \int_{\Omega} \left[c^2 |u|^2 + |\nabla u|^2 + \left| \frac{\partial u}{\partial t} \right|^2 \right] dx. \tag{2}$$

By using of the Fourier Analysis technique, one can obtain the following estimate (see [9,10]):

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$$\left|u(x,t)\right| \leqslant C|t|^{-N/2} \sum_{|\alpha|+j\leqslant(N+3)/2,\ j=0,1} \int \left|\frac{\partial^{|\alpha|}}{(\partial y)^{\alpha}} u_j(y)\right| dy, \tag{3}$$

for $|t|$ large enough and some constant $C > 0$ independent of u_0, u_1. Here $\alpha = (\alpha_1, \ldots, \alpha_N)$ is a multi-index of order $|\alpha|$ and $\frac{\partial^{|\alpha|}}{(\partial y)^{\alpha}}$ the corresponding usual partial derivative.

Let Ω be a bounded domain of $\mathbb{R}^N$ and m a positive integer. We denote by $H^m(\Omega)$ the standard Sobolev space with its usual norm $\|\cdot\|_{H^m(\Omega)}$ (see [3]). Assuming $u_0, u_1 \in C_0^\infty(\Omega)$, from (3) we obtain

$$\left|u(x,t)\right|^2 \leqslant \frac{K}{t^N}\left\{\|u_0\|^2_{H^m(\Omega)} + \|u_1\|^2_{H^{m-1}(\Omega)}\right\} \tag{4}$$

for large enough $|t|$ and every integer $m \geqslant \frac{N+3}{2} \geqslant 2$. Estimates like (4) are useful to study boundary control for hyperbolic equations in bounded domains (see [15]). In particular, estimate (4) allows us to extend the previous controllability result for the homogeneous wave equation (see [17]) to Eq. (1).

In the last four decades much has been done on boundary control for the wave equation. It is worth mentioning the surveys [18] and [11]. In [18], exact boundary controllability for the wave equation with smooth initial data is treated. In [11], J.-L. Lions uses its Hilbert Uniqueness Method (HUM) and treats control problem with initial data $L^2(\Omega) \times H^{-1}(\Omega)$. In both works, only domains with smooth boundary are considered. Control for the wave equation in nonsmooth domains was first studied in [7]. There, Grisvard uses HUM to study the wave equation in polygons and polyhedrons. In [12] and [1] Russell's approach has been used to study control for wave equation with finite energy initial state in nonsmooth domains. The use of HUM to solve the problems treated here would require a large amount of work as can be seen in [8].

In the present work we prove that given any bounded domain Ω, there exists a constant $K > 0$, such that for $u_0, u_1 \in C_0^\infty(\Omega)$ the solution of (1) satisfies

$$|u|^2 + |\nabla u|^2 + \left|\frac{\partial u}{\partial t}\right|^2 \leqslant \frac{K}{t^N}\left\{\|u_0\|^2_{H^1(\Omega)} + \|u_1\|^2_{L^2(\Omega)}\right\}, \tag{5}$$

for every $x \in \Omega$ and $t > 0$ sufficiently large. After, we indicate how to use (5) to obtain exact boundary controllability for finite energy solutions of Klein–Gordon equation in general bounded domains of $\mathbb{R}^N$. The method used here, along with appropriate trace theorems allows us to solve control problems with several types of boundary controls. In particular, for a bounded and piecewise smooth domain U of $\mathbb{R}^N$, we prove that initial state in $H^1(U) \times L^2(U)$ is driven to null state in finite time $T > 0$, by the action of square integrable Neumann control over the boundary ∂U.

It is worth noticing that estimate (4) for $m = 1$ and $N = 1, 2, 3$ has been known for a long time (see [13,14]). The estimate (5) for $N = 2$ was used in [2] to study a control problem for a system of coupled wave equations.

To the best of our knowledge, the literature does not provide an estimate like (5) for all space dimensions. In order to obtain the estimate (5) we use the explicit formula for the solution u of the Cauchy problem (1), which can be deduced following [16]. We consider separately the cases when N is odd, and when N is even, since the representation of u changes with the parity of N. For N odd, the formula of u involves the Bessel's functions of first kind. In that case, to obtain (5) we use some well known facts about the asymptotic behavior of those functions.

The paper is organized as follows. In Section 2 we announce the main result: Theorem 2.1, and apply it to the control problem. In Section 3 we present some preliminaries results, and Section 4 is dedicated to the proof of Theorem 2.1.