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Length-expanding Lipschitz maps on totally regular continua

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ABSTRACT

The tent map is an elementary example of an interval map possessing many interesting properties, such as dense periodicity, exactness, Lipschitzness and a kind of length-expansiveness. It is often used in constructions of dynamical systems on the interval/trees/graphs. The purpose of the present paper is to construct, on totally regular continua (i.e. on topologically rectifiable curves), maps sharing some typical properties with the tent map. These maps will be called length-expanding Lipschitz maps, briefly LEL maps. We show that every totally regular continuum endowed with a suitable metric admits a LEL map. As an application we obtain that every non-degenerate totally regular continuum admits an exactly Devaney chaotic map with finite entropy and the specification property.

1. Introduction

The tent map is the piecewise linear map f on the interval $I = [0,1]$ given by $f(x) = 2\min\{x, 1-x\}$. The properties of this map, conjugate to the full logistic map, include Lipschitzness, length-expansiveness (in a sense that it doubles the length of every subinterval J of I not containing $1/2$), exactness, specification, finite positive topological entropy and dense periodicity, just to name a few. This map, together with "generalized" tent maps, i.e. piecewise linear continuous maps $f_k : I \to I$ $(k \geqslant 3)$ fixing 0 and mapping linearly every interval $[(i-1)/k, i/k]$ onto I, are frequently used in dynamics. Usefulness of these maps lies in the fact that on one hand they are very simple (and so we have easy explicit formulae for iterates, periodic points, horseshoes, etc.) and on the other hand they are very "powerful". They are often used in constructions of systems on the interval/trees/graphs with special properties. For example, it is known that to construct a transitive map on the unit interval with the smallest possible topological entropy, one can define $g : I \to I$ in such a way that $1/2$ is a fixed point, g maps linearly $I_0 = [0, 1/2]$ onto $I_1 = [1/2, 1]$ and $g|_{I_1} : I_1 \to I_0$ is "tent-like". Analogously one can define a transitive map with the smallest possible entropy $(1/n)\log 2$ on any n-star S_n $(n \geqslant 3)$, see [3]; the map fixes the branch point of S_n, maps cyclically each branch to the next one, all but one linearly and the remaining one in a "tent-like" way.

Unfortunately, when one wants to construct a map with given properties on curves more general than graphs, he/she faces the problem that no direct analogue of the tent map on such curves is known. Take e.g. the ω-star X, which is a very simple dendrite defined as an infinite wedge of arcs. A construction of a transitive finite entropy map on X is much more complicated than on n-stars and, as far as we know, no such construction has been available in literature. The only result in this direction known to us is the theorem of Agronsky and Ceder [1] stating that any finite-dimensional Peano continuum (hence also the ω-star) admits a transitive map; however, the proof does not say anything about the entropy of the map.

The purpose of the present paper is to construct, on continua more general than graphs, a family of maps sharing some typical properties with the tent map. Since the key property of these maps will deal with the *length* (Hausdorff

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one-dimensional measure) of subcontinua and their images, the natural class of spaces to consider is the class of *rectifiable curves*, i.e. continua of finite length. Topologically they coincide with the class of totally regular continua. Recall that a continuum X is *totally regular* if for every point $x \in X$ and every countable set $P \subseteq X$ there is a basis of neighborhoods of x with finite boundary not intersecting P. This notion was introduced in [20], but the class of these continua was studied a long time before, see e.g. [22,8,10,9]. For more details on totally regular continua see Section 2.4.

Before stating the main results of the paper we need to introduce the notion of a length-expanding Lipschitz map. Let X be a non-degenerate totally regular continuum. We say that a family $\mathcal{C}$ of non-degenerate subcontinua of X is *dense* if every nonempty open set in X contains a member of $\mathcal{C}$. Recall that a map $f:(X,d)\to(X',d')$ between metric spaces is *Lipschitz-L* if $d'(f(x),f(y))\leqslant L\cdot d(x,y)$ for every $x,y\in X$. For a metric space (X,d), the Hausdorff one-dimensional measure is denoted by $\mathcal{H}^1_d$.

Definition A. Let $X=(X,d)$, $X'=(X',d')$ be non-degenerate (totally regular) continua of finite length and let $\mathcal{C}$, $\mathcal{C}'$ be dense systems of subcontinua of X, X', respectively. We say that a continuous map $f:X\to X'$ is *length-expanding* with respect to $\mathcal{C}$, $\mathcal{C}'$ if there exists $\varrho>1$ (called *length-expansivity constant* of f) such that, for every $C\in\mathcal{C}$, $f(C)\in\mathcal{C}'$ and

$$\text{if } f(C)\neq X' \quad \text{then } \mathcal{H}^1_{d'}\big(f(C)\big)\geqslant\varrho\cdot\mathcal{H}^1_d(C).$$

Moreover, if f is surjective and Lipschitz-L we say that $f:(X,d,\mathcal{C})\to(X',d',\mathcal{C}')$ is (ϱ,L)-*length-expanding Lipschitz*. Sometimes we briefly say that f is (ϱ,L)-*LEL* or only *LEL*. On the other hand, when we wish to be more precise, we say that f is $(\mathcal{C},\mathcal{C}',\varrho,L)$-*LEL*.

A few comments are necessary. Assume that $f:(X,d,\mathcal{C})\to(X',d',\mathcal{C}')$ is (ϱ,L)-LEL and denote by $\mathcal{C}_X$ and $\mathcal{C}_{X'}$ the systems of *all* non-degenerate subcontinua of X and X', respectively. Obviously, then also $f:(X,d,\mathcal{C})\to(X',d',\mathcal{C}_{X'})$ is (ϱ,L)-LEL. However, one cannot claim that $f:(X,d,\mathcal{C}_X)\to(X',d',\mathcal{C}')$ is (ϱ,L)-LEL. In fact, for some spaces (X,d), (X',d') there is no LEL map $f:(X,d,\mathcal{C}_X)\to(X',d',\mathcal{C}_{X'})$. For instance this is the case when X is the ω-star and $X'=I$. To show this, suppose that there is a (ϱ,L)-LEL map $f:(X,d,\mathcal{C}_X)\to(X',d',\mathcal{C}_{X'})$. Take $k\in\mathbb{N}$ such that $\varrho>L/k$ and find a k-star C in X such that every edge of C is mapped onto the same proper subinterval of X'. Then $\mathcal{H}^1_{d'}(f(C))\leqslant(L/k)\cdot\mathcal{H}^1_d(C)<\varrho\cdot\mathcal{H}^1_d(C)$, a contradiction.

Our first result says that in the special case when $X=X'$ and $\mathcal{C}=\mathcal{C}'$, LEL maps have interesting dynamical properties. (For the definitions of the corresponding notions, see Section 2.)

Proposition B. *Let $f:(X,d,\mathcal{C})\to(X,d,\mathcal{C})$ be a LEL map. Then f is exact and has finite positive entropy. Moreover, if f is the composition $\varphi\circ\psi$ of some maps $\psi:X\to I$ and $\varphi:I\to X$, then f has the specification property and so it is exactly Devaney chaotic.*

The above mentioned tent-like maps $f_k:I\to I$ (where $k\geqslant3$ and I is endowed with the Euclidean metric d_I) are $(\mathcal{C}_I,\mathcal{C}_I,k/2,k)$-LEL, where $\mathcal{C}_I$ is the system of all non-degenerate closed subintervals of I. Here $k\geqslant3$ because the classical tent map f_2 is not $(\mathcal{C}_I,\mathcal{C}_I,\varrho,L)$-LEL for any $\varrho>1$ and any L. However, it becomes $(\mathcal{C}_I,\mathcal{C}_I,\varrho,L)$-LEL (for some $\varrho>1$ and L) after a slight change of the metric. One can easily construct examples of LEL maps between arbitrary graphs, even in the form of the composition $\varphi\circ\psi$ as in Proposition B; one can use e.g. the maps from [2, Lemma 3.6]. Further, for a given continuum (X,d) of finite length, one can often find $\mathcal{C}$, $\mathcal{C}'$ and construct LEL-maps $\varphi:(I,d_I,\mathcal{C}_I)\to(X,d,\mathcal{C})$ and $\psi:(X,d,\mathcal{C}')\to(I,d_I,\mathcal{C}_I)$. However, it is not so easy to obtain $\mathcal{C}'\supseteq\mathcal{C}$; this inclusion is desirable since then also the composition $\psi\circ\varphi$ is LEL (see Lemma 9).

Our main results, the proofs of which were inspired by [1] and [6], assert that such LEL maps can always be found provided we allow to change the metric on X (the new metric still being compatible with the topology). Recall that a metric d on X is *convex* if for every $x,y\in X$ there is $z\in X$ such that $d(x,z)=d(z,y)=d(x,y)/2$. For two points $a,b\in X$ of a continuum X, $\mathrm{Cut}_X(a,b)$ denotes the set of points $x\in X$ such that a,b belong to different components of $X\setminus\{x\}$.

Theorem C. *For every non-degenerate totally regular continuum X and every $a,b\in X$ we can find a convex metric $d=d_{X,a,b}$ on X and Lipschitz surjections $\varphi_{X,a,b}:I\to X$, $\psi_{X,a,b}:X\to I$ with the following properties:*

(a) $\mathcal{H}^1_d(X)=1$;
(b) *the system $\mathcal{C}=\mathcal{C}_{X,a,b}=\{\varphi_{X,a,b}(J)\colon J$ is a closed subinterval of $I\}$ is a dense system of subcontinua of X;*
(c) *for every $\varrho>1$ there are a constant L_ϱ (depending only on ϱ) and (ϱ,L_ϱ)-LEL maps*

$$\varphi:(I,d_I,\mathcal{C}_I)\to(X,d,\mathcal{C}) \quad \text{and} \quad \psi:(X,d,\mathcal{C})\to(I,d_I,\mathcal{C}_I)$$

with $\varphi(0)=a$, $\varphi(1)=b$, $\psi(a)=0$ and such that $\varphi=\varphi_{X,a,b}\circ f_k$, $\psi=f_l\circ\psi_{X,a,b}$ for some $k,l\geqslant3$.

Moreover, if $\mathrm{Cut}_X(a,b)$ is uncountable, d, φ, ψ can be assumed to satisfy

(d) *$d(a,b)>1/2$ and $\psi(b)=1$.*