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Periodic solutions in an array of coupled FitzHugh–Nagumo cells

Isabel Salgado Labouriau [a,*], Adrian C. Murza [b]

[a] Centro de Matemática da Universidade do Porto, Rua do Campo Alegre 687, 4169-007 Porto, Portugal
[b] Departamento de Matemática, Faculdade de Ciências, Universidade do Porto, Rua do Campo Alegre 687, 4169-007 Porto, Portugal

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ABSTRACT

We analyse the dynamics of an array of N^2 identical cells coupled in the shape of a torus. Each cell is a 2-dimensional ordinary differential equation of FitzHugh–Nagumo type and the total system is $\mathbb{Z}_N \times \mathbb{Z}_N$-symmetric. The possible patterns of oscillation, compatible with the symmetry, are described. The types of patterns that effectively arise through Hopf bifurcation are shown to depend on the signs of the coupling constants, under conditions ensuring that the equations have only one equilibrium state.

1. Introduction

Hopf bifurcation has been intensively studied in equivariant dynamical systems in the recent years from both theoretical and applied points of view. Stability of equilibria, synchronisation of periodic solution and in general oscillation patterns, stability of the limit cycles that arise at the bifurcation point are among the phenomena whose analysis is related to the Hopf bifurcation in these systems. Periodic solutions arising in systems with dihedral group symmetry were studied by Golubitsky et al. [10] and Swift [8], Dias and Rodrigues [2] dealt with the symmetric group, Sigrist [12] with the orthogonal group, to cite just a few of them. Dias et al. [1] studied periodic solutions in coupled cell systems with internal symmetries, while Dionne extended the analysis to Hopf bifurcation in equivariant dynamical systems with wreath product [3] and direct product groups [4]. The general theory of patterns of oscillation arising in systems with abelian symmetry was developed by Filipsky and Golubitsky [5]. The dynamical behaviour of 1-dimensional ordinary differential equations coupled in a square array of arbitrary size $(2N)^2$ with the symmetry $\mathbf{D}_4 \dot{+} (\mathbb{Z}_N \times \mathbb{Z}_N)$, was studied by Gillis and Golubitsky [6].

In this paper we use a similar idea to that of [6] to describe arrays of N^2 cells where each cell is represented by a subsystem that is a 2-dimensional differential equation of FitzHugh–Nagumo type. We are interested in the periodic solutions arising at a first Hopf bifurcation from the fully synchronised equilibrium. To each equation in the array we add a coupling term that describes how each cell is affected by its neighbours. The coupling may be associative, when it tends to reduce the difference between consecutive cells, or dissociative, when differences are increased. For associative coupling we find, not surprisingly, bifurcation into a stable periodic solution where all the cells are synchronised with identical behaviour.

When the coupling is dissociative in either one or both directions, the first Hopf bifurcation gives rise to rings of N fully synchronised cells. All the rings oscillate with the same period, with a $\frac{1}{N}$-period phase shift between rings. When there is one direction of associative coupling, the synchrony rings are organised along it. Dissociative coupling in both directions

* Corresponding author.

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yields rings organised along the diagonal. The stability of these periodic solutions was studied numerically and they were found to be unstable for small numbers of cells, stability starts to appear at $N \geqslant 11$.

For all types of coupling, there are further Hopf bifurcations, but these necessarily yield unstable solutions.

This paper is organised as follows. The equations are presented in Section 2 together with their symmetries. Details about the action of the symmetry group $\mathbb{Z}_N \times \mathbb{Z}_N$ are summarised in Section 3: we identify the $\mathbb{Z}_N \times \mathbb{Z}_N$-irreducible subspaces of $\mathbb{R}^{2N^2}$; the isotypic components; isotropy subgroups and their fixed point subspaces for this action. This allows, in Section 4, the study of the Hopf bifurcation with symmetry $\mathbb{Z}_N \times \mathbb{Z}_N$, applying the abelian Hopf bifurcation theorem [5] to identify the symmetries of the branch of small-amplitude periodic solutions that may bifurcate from equilibria. In Section 5 we derive the explicit expression of the $2N^2$ eigenvectors and eigenvalues of the system linearised about the origin. Next, in Section 6 we perform a detailed analysis on the Hopf bifurcation by setting a parameter c to zero. In this case the FitzHugh–Nagumo equation reduces to a Van der Pol-like equation. Finally, in Section 7, we characterise the bifurcation conditions for $c > 0$ small.

2. Dynamics of FitzHugh–Nagumo coupled in a torus and its symmetries

The building-blocks of our square array are the following 2-dimensional ordinary differential equations of FitzHugh–Nagumo (FHN) type

$$\begin{aligned}
&\dot{x} = x(a-x)(x-1) - y = f_1(x, y),\\
&\dot{y} = bx - cy = f_2(x, y)
\end{aligned} \tag{1}$$

where $a, b, c \geqslant 0$. Consider a system of N^2 such equations, coupled as a discrete torus:

$$\begin{aligned}
&\dot{x}_{\alpha,\beta} = x_{\alpha,\beta}(a - x_{\alpha,\beta})(x_{\alpha,\beta} - 1) - y_{\alpha,\beta} + \gamma(x_{\alpha,\beta} - x_{\alpha+1,\beta}) + \delta(x_{\alpha,\beta} - x_{\alpha,\beta+1}),\\
&\dot{y}_{\alpha,\beta} = bx_{\alpha,\beta} - cy_{\alpha,\beta}
\end{aligned} \tag{2}$$

where $\gamma \neq \delta$ and $1 \leqslant \alpha \leqslant N$, $1 \leqslant \beta \leqslant N$, with both α and β computed (mod N). When either γ or δ is negative, we say that the coupling is *associative*: the coupling term tends to reduce the difference to the neighbouring cell, otherwise we say the coupling is *dissociative*. We restrict ourselves to the case where $N \geqslant 3$ is prime.

The coupling structure in (2) is similar, but not identical, to the general case studied by Gillis and Golubitsky [6]. There are two main differences: first, they consider an arbitrary *even* number of cells, whereas we study a prime number of cells. The second difference is that cells in [6] are bidirectionally coupled, and the coupling in (2) is unidirectional. These differences will be reflected in the symmetries of (2).

The first step in our analysis consists in describing the symmetries of (2). Our phase space is

$$\mathbb{R}^{2N^2} = \{(x_{\alpha,\beta}, y_{\alpha,\beta}) \mid 1 \leqslant \alpha, \beta \leqslant N,\ x_{\alpha,\beta}, y_{\alpha,\beta} \in \mathbb{R}\}$$

and (2) is equivariant under the cyclic permutation of the columns in the squared array:

$$\gamma_1(x_{1,\beta}, \ldots, x_{N,\beta}; y_{1,\beta}, \ldots, y_{N,\beta}) = (x_{2,\beta}, \ldots, x_{N,\beta}, x_{1,\beta}; y_{2,\beta}, \ldots, y_{N,\beta}, y_{1,\beta}) \tag{3}$$

as well as under the cyclic permutation of the rows in the squared array:

$$\gamma_2(x_{\alpha,1}, \ldots, x_{\alpha,N}; y_{\alpha,1}, \ldots, y_{\alpha,N}) = (x_{\alpha,2}, \ldots, x_{\alpha,N}, x_{\alpha,1}; y_{\alpha,2}, \ldots, y_{\alpha,N}, y_{\alpha,1}). \tag{4}$$

Thus, the symmetry group of (2) is the group generated by γ_1 and γ_2 denoted $\mathbb{Z}_N \times \mathbb{Z}_N = \langle \gamma_1, \gamma_2 \rangle$. Note that, since the coupling in (2) is unidirectional and since the coupling constants γ and δ are not necessarily equal, there is no additional symmetry, like the $\mathbf{D}_4$ in [6]. Indeed, we will show that the case $\gamma = \delta$ is degenerate. We will use the notation $\gamma_1^r \cdot \gamma_2^s \in \mathbb{Z}_N \times \mathbb{Z}_N$ as $(r, s) = \gamma_1^r \cdot \gamma_2^s$. We refer to the system (2) in an abbreviated form as either $\dot{z} = f(z)$, $z = (x_{\alpha,\beta}, y_{\alpha,\beta})$ or $\dot{z} = f(z, \lambda)$ where $\lambda \in \mathbb{R}$ is a bifurcation parameter to be specified later. The compact Lie group $\mathbb{Z}_N \times \mathbb{Z}_N$ acts linearly on $\mathbb{R}^{2N^2}$ and f commutes with it (or is $\mathbb{Z}_N \times \mathbb{Z}_N$-equivariant).

We start by recalling some definitions from [10] adapted to our case.

The isotropy subgroup Σ_z of $\mathbb{Z}_N \times \mathbb{Z}_N$ at a point $z \in \mathbb{R}^{N^2}$ is defined to be

$$\Sigma_z = \{(r, s) \in \mathbb{Z}_N \times \mathbb{Z}_N \colon (r, s) \cdot z = z\}.$$

Moreover, the fixed point subspace of a subgroup $\Sigma \in \mathbb{Z}_N \times \mathbb{Z}_N$ is

$$\mathrm{Fix}(\Sigma) = \{z \in \mathbb{R}^{2N^2} \colon (r, s) \cdot z = z,\ \forall (r, s) \in \Sigma\}$$

and $f(\mathrm{Fix}(\Sigma)) \subseteq \mathrm{Fix}(\Sigma)$.