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Metric regularity of composition set-valued mappings: Metric setting and coderivative conditions

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ABSTRACT

The paper concerns a new method to obtain a proof of the openness at linear rate/metric regularity of composite set-valued maps on metric spaces by the unification and refinement of several methods developed somehow separately in several works of the authors. In fact, this work is a synthesis and a precise specialization to a general situation of some techniques explored in the last years in the literature. In turn, these techniques are based on several important concepts (like error bounds, lower semicontinuous envelope of a set-valued map, local composition stability of multifunctions) and allow us to obtain two new proofs of a recent result having deep roots in the topic of regularity of mappings. Moreover, we make clear the idea that it is possible to use (co)derivative conditions as tools of proof for openness results in very general situations.

1. Introduction

The property of metric regularity has its origins in the open mapping principle for linear operators obtained in the 1930s by Banach and Schauder, and is one of the three basic and crucial principles of functional analysis, having various applications in many branches of mathematics. Later on, this principle was reinterpreted and generalized in two classical results: the tangent space theorem of Lyusternik [44] and the surjection theorem of Graves [28]. The next decisive step in this history was the extension of the Banach–Schauder principle to the case of set-valued maps with closed and convex graph given independently by Ursescu in [63] and Robinson in [59] (the celebrated Robinson–Ursescu Theorem). Moreover, it was observed in Dmitruk, Milyutin, and Osmolovsky [14] that the original proof of Lyusternik from [44] is applicable to a much more general setting: the sum of a covering at a rate $a > 0$ single-valued mapping and a Lipschitz one with a Lipschitz constant $b < a$ is covering at the rate $a - b$. Another remarkable insight given in the mentioned paper is that it clearly emphasizes the metric nature of openness and regularity properties. Afterwards, in 1996, Ursescu [64] was the first to obtain a fully set-valued extension of the above results, in the setting of Banach spaces. On the tracks of [14], the important work of Ioffe [30] made the crucial observation that the Lyusternik iteration process can be successfully used when the original space is a complete metric space and the image space has a linear structure with shift-invariant metric, in order to prove the preservation of regularity under single-valued Lipschitz perturbations. Detailed studies on the case of the sum of a metrically regular set-valued mapping and a single-valued Lipschitz map appear, more recently, in works by Dontchev, Lewis, and Rockafellar [19], Dontchev, and Lewis [18], Arutyunov [2,3], Mordukhovich [46]. For a detailed account

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for the whole topic of regularity properties of mappings, as well as various applications the reader is referred to the books or works of many researchers: [5–13,15,19,20,27,29–31,33–40,44,46,48,49,52,53,55,56,60,62,65].

In the last years, the study of openness at linear rate (or equivalently metric regularity) of multifunctions obtained as operations with set-valued maps has received a new impetus coming from at least three connected issues: the link between Lyusternik–Graves type theorems and fixed point assertions [2,16,17], the growing interest to generalized forms of compositions [31,25] and the new developments of metric regularity results obtained under assumptions based on generalized differentiation calculus and especially on coderivative conditions [50,22].

Included in this stream, the present paper concerns a new method to obtain a proof of the openness at linear rate/metric regularity of composite set-valued maps on metric spaces by the unification and refinement of several methods developed somehow separately in several works of the authors: [52,54,50,51,23,22,25,26]. In some sense, this is a synthesis and a precise specialization to a general situation of some techniques explored in the quoted papers. In turn these techniques are based on several important concepts (like error bounds, strong slope associated to a function, lower semicontinuous envelope of a set-valued map, local composition stability of multifunctions) and allow us to obtain two new proofs of a recent result having deep roots in the literature on the topic of regularity of mappings.

More precisely, the corner stones that this work rely on are mainly the results in [52] on the error bounds for a nonlinear variational system, the main result in [26] concerning the openness of a composite multifunction, and also the coderivative conditions for metric regularity as these appear in [50,22].

The main result of the paper (Theorem 3.8) is prepared by several propositions being of interest on their own. In that main result one obtains, under some already standard (hence expected) assumptions (see [25,26]), the openness of an auxiliary multifunction associated to a composition set-valued map, and on this basis, a result of openness around the reference point for the considered composition. We want to emphasize here two main points both of them revealing the novelty and the relevance of our work. Firstly, the conclusion is significantly richer than the corresponding conclusions of the main results in [51] (from the point of view of the generality of set-valued operations), [25,26] (from the point of view of the type of openness). Secondly, the proof is obtained using Ekeland Variational Principle (EVP, for short), a fact that answers the following question: how to get proofs for openness results (and, also, for coincidence/fixed points results), on complete metric spaces, using EVP and not arguing by contradiction. In our knowledge (see, for instance, [64,16,25]), in many cases, the proofs relying on EVP are made on normed vector spaces, and reasoning by contradiction. The supplemental structure of the space (i.e., its linear structure, but also the norm), which seems at first glance a little surprising, it is used essentially in the construction of the contradiction. In this work, by the analysis of some ideas spread in different articles (see [54,50,22]), we reached the conclusion that in order to obtain a proof in the sense discussed before, one must apply EVP to the lower semicontinuous envelope of a certain distance function, by the appropriate choice of an auxiliary multifunction involved in the construction of this envelope. As a consequence, by combining and extending some techniques from the quoted articles, we are able to give here a complete and positive (i.e., not arguing by contradiction) proof, based on EVP, for the metric regularity of set-valued mappings of composition type. Moreover, in this way, we bring more light on the links between several tools used in getting regularity results for multifunctions.

As a by-product of the main result, a coincidence/fixed points assertion is obtained, a fact that contributes to a discussion on this subject initiated in [2–4] and continued in [16,17,25]. Furthermore, the important role of the assertions before the main result is again emphasized, as the (immediate) proof of the fixed point assertion relies on the appropriate application of one of them and of the main result (Theorem 3.8).

The last section deals with coderivative conditions for openness of composite mapping. Here we reconsider several ideas in [51] and [24] and we employ a calculus rule for the Fréchet normal cone to the intersection of sets, passing through the concept of alliedness introduced and studied by Penot and his coauthors [57,43]. Finally, as an interesting fact which makes the link to the preceding section, we prove that one can obtain, on Asplund spaces, the conclusion of the main result of the paper by the use of the coderivative condition previously developed.

2. Preliminaries

This section contains some basic definitions and results used in the sequel. In what follows, we suppose that the involved spaces are metric spaces, unless otherwise stated. In this setting, $B(x,r)$ and $\overline{B}(x,r)$ denote the open and the closed ball with center x and radius r, respectively. On a product space we usually take the sum metric; when we choose another metric, this will be stated explicitly. If $x \in X$ and $A \subset X$, one defines the distance from x to A as $d(x,A) := \inf\{d(x,a) \mid a \in A\}$. As usual, we use the convention $d(x,\emptyset) = \infty$. The excess from a set A to a set B is defined as $e(A,B) := \sup\{d(a,B) \mid a \in A\}$, and the distance between A and B is given by $d(A,B) := \inf\{d(a,b) \mid a \in A, b \in B\}$. For a non-empty set $A \subset X$ we put $\operatorname{cl} A$ for its topological closure. One says that a set A is locally complete (closed) if there exists $r > 0$ such that $A \cap \overline{B}(x,r)$ is complete (closed). The symbol $\mathcal{V}(x)$ stands for the system of neighborhoods of x.

Let X, Y, Z, P be metric spaces. For a multifunction $F : X \rightrightarrows Y$, the graph of F is the set $\operatorname{Gr} F := \{(x,y) \in X \times Y \mid y \in F(x)\}$. If $A \subset X$, then $F(A) := \bigcup_{x \in A} F(x)$. The inverse set-valued map of F is $F^{-1} : Y \rightrightarrows X$ given by $F^{-1}(y) = \{x \in X \mid y \in F(x)\}$. If $F_1 : X \rightrightarrows Y$, $F_2 : X \rightrightarrows Z$, we define the set-valued map $(F_1, F_2) : X \rightrightarrows Y \times Z$ by $(F_1, F_2)(x) := F_1(x) \times F_2(x)$. For a parametric multifunction $F : X \times P \rightrightarrows Y$, we use the notations: $F_p(\cdot) := F(\cdot, p)$ and $F_x(\cdot) := F(x, \cdot)$.