J. Math. Anal. Appl. 412 (2014) 99–102

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Unital invertibility-preserving linear maps into matrix spaces

Constantin Costara

Faculty of Mathematics and Informatics, Ovidius University, Mamaia Boul. 124, 900527, Constanța, Romania

ARTICLE INFO

Article history:
Received 21 May 2013
Available online 21 October 2013
Submitted by J.D.M. Wright

Keywords:
Invertibility preserving
Linear maps
Spectrum function
Continuity

ABSTRACT

We characterize unital invertibility-preserving linear maps from a complex, unital Banach algebra $\mathcal{A}$ into $\mathcal{M}_n$, with no continuity assumption on them.

1. Introduction and statement of results

Let $\mathcal{A}$ be a unital Banach algebra over the complex field $\mathbf{C}$, and let $\mathbf{1} \in \mathcal{A}$ be its identity. For $x \in \mathcal{A}$, let $\sigma(x) \subseteq \mathbf{C}$ be its spectrum and $\rho(x)$ the spectral radius of x, that is the maximum modulus of $\sigma(x)$. The definition of the spectrum shows that if $\chi : \mathcal{A} \to \mathbf{C}$ is a character of $\mathcal{A}$ (that is, a non-zero linear and multiplicative functional), then $\chi(x) \in \sigma(x)$ for each x in $\mathcal{A}$. In particular χ is also unital, that is $\chi(\mathbf{1}) = 1$. Now if χ is a unital, linear functional on $\mathcal{A}$, then $\chi(x)$ belongs to the spectrum of x for each x in $\mathcal{A}$ if, and only if, χ sends invertible elements of $\mathcal{A}$ into invertible (non-zero) elements of $\mathbf{C}$. Gleason [5] and Kahane and Żelazko [7] proved that in the class of unital, linear functionals, the invertibility-preserving property characterizes the multiplicative ones. This was further generalized by Kowalski and Slodkowski in [8], where they proved that if $f : \mathcal{A} \to \mathbf{C}$ with $f(0) = 0$ satisfies $f(x) - f(y) \in \sigma(x - y)$ for every $x, y \in \mathcal{A}$, then f is automatically linear, and therefore also multiplicative.

Another way to generalize the characterization of multiplicative functionals given by Gleason, Kahane and Żelazko is to replace $\mathbf{C}$ with the space of $n \times n$ complex matrices $\mathcal{M}_n$, for some $n \in \mathbf{N}$. Once more, if $\varphi : \mathcal{A} \to \mathcal{M}_n$ is linear such that $\varphi(\mathbf{1}) = I_n$, the $n \times n$ unit matrix, then φ preserves invertibility if, and only if,

$$\sigma\big(\varphi(x)\big) \subseteq \sigma(x) \quad (x \in \mathcal{A}). \tag{1}$$

In the case $n = 1$ the continuity of such φ is automatic, since (1) implies $|\varphi(x)| \leqslant \rho(x) \leqslant \|x\|$ for $x \in \mathcal{A}$. If $n > 1$, to have continuity for such a map φ we need further assumptions on it. For example, if φ is surjective, then φ is continuous (see, e.g., [1, p. 13] or [2, Theorem 5.5.2]). In fact, the linear, unital, surjective and invertibility-preserving maps into $\mathcal{M}_n$ are nothing but Jordan morphisms.

Theorem 1. *(See [1, Theorem 1].) Let φ be a linear, unital and invertibility-preserving map from $\mathcal{A}$ onto $\mathcal{M}_n$. Then φ is an algebra morphism or an algebra antimorphism.*

E-mail address: cdcostara@univ-ovidius.ro.

0022-247X/$ – see front matter
http://dx.doi.org/10.1016/j.jmaa.2013.10.045

The result of Aupetit was generalized by Christensen in [3], by removing the surjectivity assumption on φ and replacing it with continuity. See also [4, Section 4] for further development of the ideas and interesting examples of non-surjective unital linear invertibility-preserving mappings into $\mathcal{M}_n$.

Theorem 2. *(See [3, Theorem 3.5].) Let φ be a continuous linear, and unital map from $\mathcal{A}$ into $\mathcal{M}_n$. Then φ is invertibility-preserving if, and only if,*

$$\operatorname{tr}\big(\varphi\big(x^k\big)\big) = \operatorname{tr}\big(\varphi(x)^k\big) \quad (k \in \mathbf{N},\ x \in \mathcal{A}). \tag{2}$$

(For $a \in \mathcal{M}_n$, by $\operatorname{tr}(a)$ we denote its usual trace.)

A unital, invertibility-preserving linear map from $\mathcal{A}$ into $\mathcal{M}_n$ is not automatically continuous. This comes from a result of Shirdareh Haghighi [6, Theorem 2.1], who obtained an explicit form for such discontinuous maps in the particular case $n = 2$. Up to a similarity, a discontinuous unital linear mapping $\varphi : \mathcal{A} \to \mathcal{M}_2$ which preserves invertibility is of the form

$$\varphi = \begin{bmatrix} \alpha & \delta \\ 0 & \beta \end{bmatrix}, \tag{3}$$

where α, β are non-zero multiplicative linear functionals on $\mathcal{A}$ and δ is a discontinuous linear functional on $\mathcal{A}$, with $\delta(e) = 0$.

For example, take $\mathcal{A}$ to be the algebra of complex-valued continuous functions on the real interval $[0, 1]$, with the uniform norm. Let α be the point evaluation at 0 and β the point evaluation at 1, and let δ be a linear functional on $\mathcal{A}$ such that $\delta(t^n) = n$ for $n = 0, 1, \ldots$. Then α and β are multiplicative, and δ is discontinuous on $\mathcal{A}$ and zero at the identity of $\mathcal{A}$. Then φ given by (3) is unital, linear and preserves invertibility, but not continuous.

Since no continuity assumption was needed in the Gleason–Kahane–Żelazko theorem, one may ask if [3, Theorem 3.5] remains true if φ is not supposed continuous. The case $n = 2$ comes from [6, Theorem 2.1]. The answer in the general case is given by the next theorem, which is the main result of this paper.

Theorem 3. *Let φ be a unital, linear mapping from $\mathcal{A}$ into $\mathcal{M}_n$. Then φ preserves invertibility if, and only if, the relations (2) hold for each $k \in \mathbf{N}$ and $x \in \mathcal{A}$.*

The map given by (3) shows that under the hypothesis of Theorem 3, the invertibility-preserving property of φ does not imply that it is necessarily continuous. The next corollary shows that we obtain continuity by taking the spectrum of φ.

Corollary 4. *Let φ be a unital linear mapping from $\mathcal{A}$ into $\mathcal{M}_n$. If φ preserves invertibility, then $\pi \circ \varphi : \mathcal{A} \to \mathbf{C}^n$ is continuous, where $\pi : \mathcal{M}_n \to \mathbf{C}^n$ is the symmetrization map, that is*

$$\pi(x) = \big(S_1(x), \ldots, S_n(x)\big) \quad (x \in \mathcal{M}_n),$$

where for $k = 1, 2, \ldots, n$, by $S_k(x)$ we have denoted the k-th symmetric function on the eigenvalues of $x \in \mathcal{M}_n$. (For example, S_1 is just the trace and S_n the determinant.)

Indeed, since (1) holds then $|\operatorname{tr}(\varphi(x))| \leqslant n\rho(x) \leqslant n\|x\|$ on $\mathcal{A}$, which means that the linear function $\operatorname{tr} \circ \varphi$ is continuous on $\mathcal{A}$. Using (2) we obtain continuity for $x \mapsto \operatorname{tr}(\varphi(x)^k)$, for each fixed k in $\mathbf{N}$. Now the classical Newton formulae imply continuity for each $S_k \circ \varphi$.

2. Proofs

If a map $\varphi : \mathcal{A} \to \mathcal{M}_n$ satisfies (1), then

$$\big|S_k\big(\varphi(x)\big)\big| \leqslant \binom{n}{k} \|x\|^k \quad (x \in \mathcal{A},\ k = 1, \ldots, n), \tag{4}$$

where for each k by $\binom{n}{k}$ we have denoted the standard binomial coefficient. In particular, we obtain continuity for each map $x \mapsto S_k(\varphi(x))$, but only at $0 \in \mathcal{A}$. If φ is also supposed linear then for the particular case $k = 1$ we obtain that $\operatorname{tr} \circ \varphi$ is continuous everywhere on $\mathcal{A}$.

In the linear case, the same inequalities involving the symmetric functions hold for invertibility-preserving maps.

Lemma 5. *Suppose $\varphi : \mathcal{A} \to \mathcal{M}_n$ is linear, unital, and preserves invertibility. Then φ satisfies (4).*

Proof. As observed in the introduction, we have that φ satisfies (1), and therefore (4) holds. $\square$

For the proof of Theorem 3 we shall need boundedness/continuity properties for more general maps than the ones given by (4). The main ingredient will be the following result, which generalize the one given by Lemma 5.