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On $O(k)$-invariant solutions of semilinear elliptic equations

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ABSTRACT

Let u be a classical solution of Dirichlet boundary value problem $-\Delta u = f(x,u)$ in Ω, where Ω is a bounded and $O(k)$-invariant domain in $\mathbb{R}^N$ $(1 \leqslant k \leqslant N)$ and the nonlinearity f is $O(k)$-invariant in the first variable and convex in the second variable. In this paper, we introduce sufficient conditions on u to ensure that u is also $O(k)$-invariant.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$, we consider the following boundary value problem

$$\begin{cases} -\Delta u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω and f are somehow symmetric in a sense to be defined later and u is a classical solution of (1.1). We would like to study the symmetry properties of u.

A classical tool to study this question is the well-known moving plane method which was introduced by Alexandrov and Serrin [12] and was successfully used by Berestycki, Gidas, Ni and Nirenberg in [4,7] to prove the radial symmetry of positive solutions to (1.1) when Ω is a ball, $f(x,s) = \overline{f}(|x|,s)$ and $\overline{f}$ is nonincreasing in the radial variable. Moreover, there are counterexamples to the symmetry of solutions if some of the hypotheses fail. For instance, see [5] for the existence of a nonradial solution in an annulus. More recently, it is proved in [8] the bifurcation of nonradial positive solutions from the radial positive solution of equation $-\Delta u = u^p + \lambda u$ in an annulus when the radii of the annulus vary or when the exponent p varies. It is also proved in [1] that if Ω is a square domain in $\mathbb{R}^2$ and $f(x,s) = w(x)s$ where w is a given positive function invariant under all (Euclidean) symmetries of the square, then (1.1) has a solution which is neither symmetric nor antisymmetric with respect to any nontrivial symmetry of the square. These examples point out that in general, the symmetry of Ω and f does not imply the symmetry of solutions.

Nevertheless, we can expect that the solutions inherit part of the symmetry of the domain at least for some types of nonlinearities or for certain types of solutions. This direction was first investigated in [10] where Pacella proved that if Ω is a ball or an annulus, $f(x,s) = \overline{f}(|x|,s)$ and $\overline{f}$ is strictly convex in s, then any solution u to (1.1) with Morse index one is axially symmetric with respect to an axis passing through the origin and nonincreasing in the polar angle from this axis. This conclusion was then expanded to solutions having Morse index $j \leqslant N$ in [11] when Ω is a ball or an annulus and in [9] when Ω is the whole $\mathbb{R}^N$ or the exterior of a ball. Afterwards, some similar results on partial symmetry for minimizers of certain variational problems were obtained in [3,13] using symmetrization techniques.

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Instead of partial symmetry, the aim of this paper is to introduce sufficient conditions on u to ensure that u fully inherits the symmetry of Ω and f. We are also interested in the general case that Ω is invariant under an orthogonal group action. Our results can be applied to cases where the moving plane method does not work. The results also improve some results in [10] when Ω is radially symmetric.

2. Preliminaries and results

In the sequel, let k be an integer such that $1 \leqslant k \leqslant N$. Let Ω be an open bounded domain in $\mathbb{R}^N$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a continuous function of class C^1 with respect to the second variable s and u be a classical $C^2(\Omega) \cap C(\overline{\Omega})$ solution of (1.1). We denote by $L_u = -\Delta - \frac{\partial}{\partial s} f(x,u)$ the linearized operator of (1.1) at u and $\lambda_k(L_u, D)$ the k-th eigenvalue of L_u in $D \subset \Omega$ with zero Dirichlet boundary conditions.

Let $\mathcal{H}$ denote the set of all open half spaces H in $\mathbb{R}^N$ such that the boundary ∂H has nonempty intersection with Ω. For $H \in \mathcal{H}$, we denote by e_H the normal vector to ∂H that is oriented towards H and r_H the reflection with respect to ∂H. We also denote $\Omega_H^+ = \Omega \cap H$ and $\Omega_H^- = \Omega \cap (\mathbb{R}^N \setminus \overline{H})$.

We denote the open ball in $\mathbb{R}^N$ of center x and radius $r > 0$ by $B(x, r)$ and define the distance between $H_1, H_2 \in \mathcal{H}$ by

$$d(H_1, H_2) = \inf\bigl\{r > 0\colon \Omega_{H_1}^+ \subset \Omega_{H_2}^+ + B(0,r),\ \Omega_{H_2}^+ \subset \Omega_{H_1}^+ + B(0,r)\bigr\}$$

then d is a metric on $\mathcal{H}$.

For the sake of simplicity, in this paper, we don't need the definitions of orthogonal groups $O(k)$, group actions and invariant sets but we will use the direct definitions of $O(k)$-invariant as follows:

Definition 2.1. A subset $\Omega \subset \mathbb{R}^N$ is called *$O(k)$-invariant* if and only if $x := (x_1, x_2, \dots, x_N) \in \Omega$ implies $(x_1', x_2', \dots, x_k', x_{k+1}, \dots, x_N) \in \Omega$ whenever $\sum_{i=1}^k x_i^2 = \sum_{i=1}^k x_i'^2$.

Definition 2.2. A function $u : \Omega \to \mathbb{R}$ is called *$O(k)$-invariant* if and only if Ω is $O(k)$-invariant and $u(x_1, x_2, \dots, x_N) = u(x_1', x_2', \dots, x_k', x_{k+1}, \dots, x_N)$ whenever $\sum_{i=1}^k x_i^2 = \sum_{i=1}^k x_i'^2$.

Remark 2.3.

(i) A subset $\Omega \subset \mathbb{R}^N$ or a function u is *radially symmetric* with respect to 0 if it is $O(N)$-invariant and *axially symmetric* with respect to the axis $\{x \in \mathbb{R}^N\colon x_1 = x_2 = \dots = x_{N-1} = 0\}$ if it is $O(N-1)$-invariant.

(ii) If Ω or u is $O(k)$-invariant then it is $O(k-1)$-invariant for $1 < k \leqslant N$ and $N \geqslant 2$.

We will use the following useful fact in later proofs:

Proposition 2.4. *A subset $\Omega \subset \mathbb{R}^N$ is $O(k)$-invariant if and only if Ω is symmetric with respect to any hyperplane K satisfying $\{0\}^k \times \mathbb{R}^{N-k} \subset K$. A similar conclusion is true for a function $u : \Omega \to \mathbb{R}^N$.*

Denote $\Omega^- = \{x \in \Omega\colon x_1 < 0\}$ and $\Omega^+ = \{x \in \Omega\colon x_1 > 0\}$. We also denote by $\lambda_1(L_u, D)$ the first eigenvalue of the linearized operator L_u at a solution u of (1.1) in $D \subset \Omega$ with zero Dirichlet boundary conditions. We will use the following result to get some partial symmetry of u. This result is Proposition 1.1 in [10] and can be proved easily using a reflection argument:

Theorem 2.5. *If Ω is symmetric with respect to the hyperplane $\{x \in \mathbb{R}^N\colon x_1 = 0\}$, $f(x,s)$ is even in x and strictly convex in s and both $\lambda_1(L_u, \Omega^-)$ and $\lambda_1(L_u, \Omega^+)$ are nonnegative, then u is symmetric with respect to the x_1-variable, that is $u(x_1, \dots, x_N) = u(-x_1, x_2, \dots, x_N)$. The same result holds if f is only convex but $\lambda_1(L_u, \Omega^-)$ and $\lambda_1(L_u, \Omega^+)$ are both positive.*

Our main results are the following theorems which will be proved in the next section. The first theorem deals with the case $2 \leqslant k \leqslant N$. The second one deals with the case $1 \leqslant k \leqslant N$ and positive solutions. Although the second theorem requires additional assumptions on f and on geometric properties of Ω, it may give us more qualitative information of u.

Theorem 2.6. *Suppose that $2 \leqslant k \leqslant N$, domain Ω is $O(k)$-invariant, $f(x,s)$ is $O(k)$-invariant in x and convex in s and u is a classical solution of (1.1). Then u is $O(k)$-invariant if $\lambda_2(L_u, \Omega) > 0$.*

Theorem 2.7. *Suppose that $1 \leqslant k \leqslant N$, domain Ω is $O(k)$-invariant, $f(x,s)$ is $O(k)$-invariant in x and strictly convex in s and u is a positive classical solution of (1.1). Furthermore, suppose that for every open half space $H \in \mathcal{H}$ satisfying $\{0\}^k \times \mathbb{R}^{N-k} \subset \partial H$, there exists a family of half spaces $\{H_\alpha \in \mathcal{H}\colon \alpha \in [0,1]\}$ such that*

(i) *$H_0 = H$ and $H_1 = \mathbb{R}^N \setminus \overline{H}$,*