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Differentiability of bizonal positive definite kernels on complex spheres

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ABSTRACT

We prove that any continuous function with domain $\{z \in \mathbb{C}\colon |z| \leqslant 1\}$ that generates a bizonal positive definite kernel on the unit sphere in $\mathbb{C}^q$, $q \geqslant 3$, is continuously differentiable in $\{z \in \mathbb{C}\colon |z| < 1\}$ up to order $q-2$, with respect to both z and $\bar{z}$. In particular, the partial derivatives of the function with respect to $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$ exist and are continuous in $\{z \in \mathbb{C}\colon |z| < 1\}$ up to the same order.

1. Introduction and notation

Let q be a positive integer and denote by Ω_{2q} the unit sphere in $\mathbb{C}^q$. A function $K : \Omega_{2q} \times \Omega_{2q} \to \mathbb{C}$ is said to be *a positive definite kernel* on Ω_{2q} if

$$\sum_{\mu=1}^{n} \sum_{\nu=1}^{n} c_\mu \bar{c}_\nu f(\zeta_\mu, \zeta_\nu) \geqslant 0, \tag{1.1}$$

whenever n is a positive integer, $\zeta_1, \zeta_2, \ldots, \zeta_n$ are distinct points on Ω_{2q} and $c_1, c_2, \ldots, c_n$ are complex numbers. If $q \geqslant 2$ and $B[0,1] := \{z \in \mathbb{C}\colon |z| \leqslant 1\}$, the kernel K is *bizonal* if

$$K(\zeta, \eta) = K'(\zeta \cdot \eta), \quad \zeta, \eta \in \Omega_{2q}, \tag{1.2}$$

for some function $K' : B[0,1] \to \mathbb{C}$. Here, "$\cdot$" denotes the usual inner product in $\mathbb{C}^q$. The case $q = 1$ does not appear in the formulation above because it does not fit (see [12]). Indeed, the usual inner product of two points in Ω_2 produces another element of Ω_2.

Continuous, bizonal and positive definite kernels on Ω_{2q} were characterized in [12] as those for which the function K' appearing in (1.2) have a double series representation of the form

$$K'(z) = \sum_{m,n=0}^{\infty} a_{m,n}^{q-2} R_{m,n}^{q-2}(z), \quad z \in B[0,1], \tag{1.3}$$

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in which $a^{q-2}_{m,n} \geqslant 0$, $m, n \in \mathbb{Z}_+$ and $\sum_{m,n=0}^{\infty} a^{q-2}_{m,n} < \infty$. The symbol $R^{q-2}_{m,n}$ stands for the disk polynomial of bi-degree (m, n) associated with the integer $q-2$ (see Section 2), a polynomial of degree m in the variable z and degree n in the variable $\bar{z}$. The continuity of each $R^{\alpha}_{m,n}$ and the convergence of the series of coefficients imply that the series representation (1.3) converges uniformly and absolutely in $B[0, 1]$. The disk polynomials are also known as generalized Zernike polynomials and, in some cases, they enter as an important tool in the analysis of problems in optical imaging and metrology-related issues (see [19,21] and references mentioned there).

The concepts and characterization above also extend to the unit sphere Ω_∞ in the complex ℓ_2. The continuous, bizonal and positive definite kernels on Ω_∞ are generated in the same way, now by a function defined by a uniform and absolutely convergent double series representation in the form

$$K'(z) = \sum_{m,n=0}^{\infty} a^{\infty}_{m,n} z^m \bar{z}^n, \quad z \in B[0, 1],$$

in which $a^{\infty}_{m,n} \geqslant 0$, $m, n \in \mathbb{Z}_+$ and $\sum_{m,n=0}^{\infty} a^{\infty}_{m,n} < \infty$ [1, p. 171]. We will write $PD(\Omega_{2q})$, $q \in \mathbb{Z}_+ \cup \{\infty\}$, to denote the class of all functions K' possessing the corresponding representation described above.

Bizonal positive definite functions on real spheres were identified in the early forties by I.J. Schoenberg. They occur with a certain frequency in statistics but also have applications in approximation theory in connection with the poisedness of certain scattered data interpolation problems, those tagged as radial basis function interpolation on spheres (see [4,5,20] and references therein). Similar interpolation problems can be formulated on complex spheres and the functions in the classes $PD(\Omega_{2q})$ play a similar role. As a matter of fact, the *strictly positive definite kernels*, that is, those for which inequality (1.1) is strict whenever n is a positive integer, $\zeta_1, \zeta_2, \ldots, \zeta_n$ are distinct points on Ω_{2q} and $c_1, c_2, \ldots, c_n$ are complex numbers not all simultaneously zero, are preferable. A continuous, bizonal and positive definite kernel K is strictly positive definite in Ω_{2q} ($K \in SPD(\Omega_{2q})$) if and only if the set $\{m - n: a^{q-2}_{m,n} > 0\}$ obtained from the representations for K' above contains infinitely many even and infinitely many odd integers (see Theorems 4.2 and 4.4 in [13]).

The issue in this paper is differentiability of functions in $PD(\Omega_{2q})$. In general, differentiability of positive definite functions (not necessarily bizonal) is related to the reproducing kernel Hilbert spaces generated by the kernel [2,6,17,23] and those are quite common in many problems from learning theory and other areas surrounding it. Another way in which the differentiability of a positive definite kernel enters is that related to the analysis of decay rates for the eigenvalues and singular values of integral operators generated by the kernel. Usually, improved decay rates demand additional assumptions on the kernel and, among them, the existence and boundedness of certain derivatives of the kernel guarantees relevant decays. Ref. [3] and others mentioned in there deal with such problem in a setting not too far from the one adopted here. The differentiability of radial positive definite functions in $\mathbb{R}^n$ goes back to Schoenberg [16] who clearly showed the smoothing effect on the function of the requirement of its positive definiteness in higher dimensional spaces. The extension of Schoenberg's result to differentiability of functions that generate bizonal positive definite kernels on real spheres was recently ratified in [24].

This is the point where we can state the main theorem to be proved in this paper. We write $B(0, 1)$ to denote the interior of $B[0, 1]$.

Theorem 1.1. *Let q be an integer at least 2. If f belongs to $PD(\Omega_{2q+2})$, then f is differentiable with respect to both z and $\bar{z}$ in $B(0, 1)$. In addition, there exist f_1, f_2, g_1 and g_2 in $PD(\Omega_{2q})$ so that*

$$\frac{\partial}{\partial z} f(z) = \left(1 - |z|^2\right)^{-1} \left[f_1(z) - f_2(z)\right], \quad z \in B(0, 1),$$

and

$$\frac{\partial}{\partial \bar{z}} f(z) = \left(1 - |z|^2\right)^{-1} \left[g_1(z) - g_2(z)\right], \quad z \in B(0, 1).$$

If f is an element of $SPD(\Omega_{2q+2})$, then the functions f_1, f_2, g_1 and g_2 can be taken from $SPD(\Omega_{2q})$.

The paper is structured as follows. In Section 2, we survey upon disk polynomials and their properties needed in the subsequent sections. Section 3 contains old and new material related to convergence of double series. In particular, it includes a criterion for the interchange of summation and differentiation symbols in the setting of the paper. Section 4 contains the actual proof of Theorem 1.1 preceded by three technical results. Consequences of Theorem 1.1 are listed at the end of the section.

2. Disk polynomials: relevant formulas

In this section we review upon disk polynomials and quote all the formulas involving them to be used ahead. Basic references for the results quoted here are [9,8,21].

Let α be a real number bigger than -1. The *disk polynomial* of bi-degree (m, n) with respect to the real number α is the function $R^{\alpha}_{m,n}$ given by the formula