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On a class of singular second-order Hamiltonian systems with infinitely many homoclinic solutions

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ABSTRACT

We show existence of infinitely many homoclinic orbits at the origin for a class of singular second-order Hamiltonian systems

$$\ddot{u} + V_u(t,u) = 0, \quad -\infty < t < \infty.$$

We use variational methods under the assumption that $V(t,u)$ satisfies the so-called "Strong-Force" condition.

0. Introduction

The search for periodic as well as homoclinic and heteroclinic solutions of Hamiltonian systems has a long and rich history. In this paper we are particularly interested in homoclinic solutions of singular second-order Hamiltonian systems with time-periodic potentials. We refer the interested reader to the book [1] of Ambrosetti and Coti Zelati for results on the literature of periodic solutions for such singular systems.

Second-order Hamiltonian systems are systems of the form

$$\ddot{u} + V_u(t,u) = 0, \quad t \in \mathbb{R},\ u \in \mathbb{R}^N. \tag{HS}$$

Loosely speaking, they are the Euler–Lagrange equations of the functional

$$I(u) = \int L(t,u,\dot{u})\,dt,$$

where the integration is taken over a finite interval $[0,T]$ or all reals $\mathbb{R}$ and the *Lagrangian* has the form

$$L(t,u,\dot{q}) = \frac{1}{2}|\dot{u}|^2 - V(t,u).$$

Clearly, when the potential $V(t,u)$ is T-periodic in t, it is natural to look for T-periodic solutions of (*HS*) as critical points of the functional $I(u)$ over a suitable space of T-periodic functions. Also, in such a case, one can look for homoclinic solutions at the origin (i.e., solutions of (*HS*) satisfying $u(t), \dot{u}(t) \to 0$) as limits of kT-periodic solutions (*subharmonic solutions*) as $k \to \infty$ (see [17]) or, alternatively, as critical points of the functional $I(u)$ over a suitable space of functions on the whole space $\mathbb{R}$ (typically, $H^1(\mathbb{R},\mathbb{R}^N)$).

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For singular systems, one assumes that $V \in C^1(\mathbb{R} \times \mathbb{R}^N \setminus S)$ and $\lim_{u \to S} |V(t,u)| = \infty$ for some $S \subset \mathbb{R}^N$. Although the study of singular systems is perhaps as old as the Kepler classical problem in mechanics,

$$\ddot{u} + \frac{u}{|u|^3} = 0$$

(and, also, the N-body problem), the interest in such problems was renewed by the pioneering papers [13] of Gordon in 1975 and [14] of Rabinowitz in 1978. In [13] the notion of Strong-Force is introduced to deal with singular problems, while in [14] the use of variational methods is brought into the study of periodic solutions of Hamiltonian systems.

The present paper is concerned with existence of homoclinic solutions for second-order Hamiltonian systems

$$\ddot{u} + V_u(t,u) = 0, \qquad (HS)$$

where $-\infty < t < \infty$, $u = (u_1, u_2, \dots, u_N) \in \mathbb{R}^N$ and the potential $V : \mathbb{R} \times \mathbb{R}^N \setminus \{q\} \to \mathbb{R}$ has a singularity $0 \neq q \in \mathbb{R}^N$. We recall that a homoclinic solution of (*HS*) is a solution such that $u(t) \in R^N \setminus \{q\}$ for all $t \in \mathbb{R}$ and

$$u(t), \dot{u}(t) \to 0 \quad \text{as } t \to \pm\infty.$$

Throughout the paper we will be considering the following assumptions on $V(t,u)$:

- (A) $V(t,u) = a(t)W(u)$, with $a \in C(\mathbb{R})$ a T-periodic function such that $a_0 \leqslant a(t) \leqslant a_\infty$ for some $a_0, a_\infty > 0$;
- (H_1) $W \in C^2(\mathbb{R}^N \setminus \{q\}, \mathbb{R})$ for some $q \in \mathbb{R}^N \setminus \{0\}$;
- (H_2) $W(0) = W_u(0) = 0$, $W(u) < W(0) = 0$ for $u \neq 0$, and $-\alpha_0 I \leqslant W_{uu}(0) \leqslant -\alpha_1 I$ for some $\alpha_0, \alpha_1 > 0$;
- (H_3) $\lim_{u \to q} W(u) = -\infty$ and there exists $U \in C^1(\mathbb{R}^N \setminus \{q\}, \mathbb{R})$ such that $\lim_{u \to q} |U(u)| = \infty$ and $W(u) \leqslant -|\nabla U(u)|^2$ for $0 < |u - q| \leqslant r$;
- (H_4) There exists $U_\infty \in C(\mathbb{R}^N \setminus B_{R_0}, \mathbb{R})$ such that $\lim_{|u| \to \infty} |U_\infty(u)| = \infty$ and $W(u) \leqslant -|\nabla U_\infty(u)|^2$ for u large.

Note that by our assumptions, W has a strict global maximum at $u = 0$ which by (H_2) is an unstable equilibrium of (*HS*). Furthermore (H_3), (H_4) concern the behavior of W close to the singularity and at infinity. In fact, (H_3) indicates that the potential W satisfies the *Strong-Force* condition mentioned earlier (used by Gordon in [13]) which governs the rate at which $W(x)$ approaches $-\infty$ as $x \to q$. A typical example is $W(x) = |x - q|^{-\alpha}$ $(\alpha \geqslant 2)$ in a neighborhood of q. On the other hand (H_4) allows W to go to zero at infinity although at a slow rate. This condition will be satisfied if, for example, $\lim_{|x| \to \infty} |x|^\beta W(x) \neq 0$ for some $\beta \in (0, 2]$.

In the case of autonomous singular Hamiltonian systems, the first result on existence of a homoclinic orbit using variational methods were obtained by Tanaka [21] under essentially the same assumptions as above. In [21] Tanaka used a minimax argument from Bahri and Rabinowitz [2] in order to get approximating solutions of the boundary value problems

$$\ddot{u} + V'(u) = 0, \quad t \in (-m, m), \qquad u(-m) = u(m) = 0$$

as critical points of the corresponding functionals, and obtained uniform estimates to show that those solutions converged weakly to a *nontrivial* homoclinic solution of (*HS*). Regarding multiplicity of homoclinics, still in the autonomous singular case, early results were obtained by Caldiroli [6], who showed existence of two homoclinic orbits, and by Bessi [4], who used Lyusternik–Schnirelmann category to prove the existence of $N - 1$ distinct homoclinics for potentials satisfying a pinching condition (see also [1] and [22] for multiplicity results in case of *smooth* Hamiltonians). Different kinds of multiplicity results were obtained in [3,7] (still for conservative systems) by exploiting the topology of $\mathbb{R}^N \setminus S$, the domain of the potential, when the set S is such that the fundamental group of $\mathbb{R}^N \setminus S$ is nontrivial.

In the case of planar autonomous systems more extensive existence and multiplicity results were obtained. Indeed, under essentially the same conditions as above with $N = 2$, Rabinowitz showed in [16] that (*HS*) has at least a pair of homoclinic solutions by exploiting the topology of the plane and minimizing the energy functional on classes of sets with a fixed winding number around the singularity q (see also [5] for results in the case of two singularities). The result in [16] was substantially improved in [8] where, using the same idea, the authors show that a nondegeneracy variational condition introduced in [16] is in fact necessary and sufficient for the minimum problem to have a solution in the class of sets with winding number greater than 1 and, therefore, proved a result on existence of infinitely many homoclinic solutions.

On the other hand, in the case of T-periodic time dependent Hamiltonians in $\mathbb{R}^N$, existence of infinitely many homoclinic orbits was obtained for *smooth* Hamiltonians by using a variational procedure due to Séré in [18] and [19] for first order systems, and in [9] and [10] for second-order equations. In the case $N = 2$, using these ideas, Rabinowitz [15] constructed infinitely many multibump homoclinic solutions for $V(t,u)$ of the form $a(t)W(u)$, with $a(t)$ being almost periodic and $W(u)$ satisfying (A), (H_1)–(H_4).

Our work here on homoclinic solutions of time-periodic singular equations was motivated by earlier works on periodic solutions of such equations as well as by [10], where homoclinic solutions in $\mathbb{R}^N$ are considered in the case of second-order *smooth* Hamiltonians. As already mentioned, the main tool in [10] is a minimax procedure of Séré which leads to construction of infinitely many multibump homoclinics. By contrast, our approach to the *singular* problem uses category theory in order to obtain infinitely many homoclinics. In fact, we consider the action functional I on the full space $\Lambda = H^1(\mathbb{R}, \mathbb{R}^N \setminus \{q\})$ and use the fact that $Cat_\Lambda(\Lambda) = \infty$ (cf. Proposition 1.7) to generate a sequence of candidates to critical levels of I: