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Weighted Bloch spaces and quadratic integrals

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ABSTRACT

Let $\mathcal{B}^\omega(B_d)$ denote the ω-weighted Bloch space in the unit ball B_d of $\mathbb{C}^d$, $d \geqslant 1$. We show that the quadratic integral

$$\int_x^1 \frac{\omega^2(t)}{t}\,dt, \quad 0<x<1,$$

governs the radial divergence and integral reverse estimates in $\mathcal{B}^\omega(B_d)$.

1. Introduction

Let $H(B_d)$ denote the space of holomorphic functions on the unit ball B_d of $\mathbb{C}^d$, $d \geqslant 1$.

1.1. Weighted Bloch spaces

Given a gauge function $\omega : (0,1] \to (0,+\infty)$, the weighted Bloch space $\mathcal{B}^\omega(B_d)$ consists of those $f \in H(B_d)$ for which

$$\|f\|_{\mathcal{B}^\omega(B_d)} = \left|f(0)\right| + \sup_{z\in B_d} \frac{|\mathcal{R}f(z)|(1-|z|)}{\omega(1-|z|)} < \infty, \tag{1}$$

where

$$\mathcal{R}f(z) = \sum_{j=1}^d z_j \frac{\partial f}{\partial z_j}(z), \quad z \in B_d,$$

is the radial derivative of f. $\mathcal{B}^\omega(B_d)$ is a Banach space with respect to the norm defined by (1). If $\omega \equiv 1$, then $\mathcal{B}^\omega(B_d)$ is the classical Bloch space $\mathcal{B}(B_d)$. Usually we suppose that the gauge function ω is increasing; hence, we have $\mathcal{B}^\omega(B_d) \subset \mathcal{B}(B_d)$.

The above notation is not completely standard: often the weight $t/\omega(t)$ is attributed to $\mathcal{B}^\omega(B_d)$.

Assuming that ω is sufficiently regular, we show in the present paper that the quadratic integral

$$I(x) = I_\omega(x) = \int_x^1 \frac{\omega^2(t)}{t}\,dt, \quad 0<x<1,$$

governs the radial divergence and integral reverse estimates in $\mathcal{B}^\omega(B_d)$. In both cases, the solutions are based on the classical Hadamard gap series.

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1.2. Radial divergence

Given $f \in H(B_d)$ and $\zeta \in \partial B_d$, we say that f has a radial limit at ζ if there exists a *finite* limit $f^*(\zeta) = \lim_{r\to 1-} f(r\zeta)$.

Let σ_d denote the normalized Lebesgue measure on the unit sphere ∂B_d. The radial convergence or divergence in $\mathcal{B}^\omega(B_d)$ is described in terms of $I(0+)$ by the following dichotomy:

Proposition 1.1. *Let $\omega : (0,1] \to (0,+\infty)$ be an increasing function.*

(i) *Let $I(0+) < \infty$. If $f \in \mathcal{B}^\omega(B_d)$, then f has radial limits σ_d-almost everywhere.*
(ii) *Let $I(0+) = \infty$ and let $\omega(t)/t^{1-\varepsilon}$ be decreasing for some $\varepsilon > 0$. Then the space $\mathcal{B}^\omega(B_d)$ contains a function with no radial limits σ_d-almost everywhere.*

Remark that the condition $I(0+) = \infty$ was previously used by Dyakonov [8] to construct a non-BMO function lying in $\mathcal{B}^\omega(B_1)$ and in all Hardy spaces $H^p(B_1)$, $0 < p < \infty$.

1.3. Reverse estimates

Given an unbounded decreasing function $v : (0,1] \to (0,+\infty)$, typical reverse estimates are obtained in the growth space $\mathcal{A}^v(B_d)$, which consists of $f \in H(B_d)$ such that $|f(z)| \leqslant Cv(1-|z|)$ for all $z \in B_d$. Namely, under appropriate restrictions on v, there exists a finite family $\{f_j\}_{j=1}^J \subset \mathcal{A}^v(B_d)$ such that

$$\big|f_1(z)\big| + \cdots + \big|f_J(z)\big| \geqslant Cv\big(1-|z|\big)$$

for all $z \in B_d$ (see, for example, [1] and references therein).

For the weighted Bloch space $\mathcal{B}^\omega(B_d)$, the following result provides integral reverse estimates related to the function $\Phi^{\frac{1}{2}}(1-|z|)$, $z \in B_d$, where

$$\Phi(x) = \Phi_\omega(x) = 1 + \int_x^1 \frac{\omega^2(t)}{t}\,dt, \quad 0 < x < 1.$$

Theorem 1.2. *Let $d \in \mathbb{N}$ and let $0 < p < \infty$. Assume that $\omega : (0,1] \to (0,+\infty)$ increases and $\omega(t)/t^{1-\varepsilon}$ decreases for some $\varepsilon > 0$. Then there exists a constant $\tau_{d,p,\omega} > 0$ and functions $F_y \in \mathcal{B}^\omega(B_d)$, $0 \leqslant y \leqslant 1$, such that $\|F_y\|_{\mathcal{B}^\omega(B_d)} \leqslant 1$ and*

$$\int_0^1 \big|F_y(z)\big|^{2p}\,dy \geqslant \tau_{d,p,\omega}\Phi^p\big(1-|z|\big) \tag{2}$$

for all $z \in B_d$.

For $\omega \equiv 1$ and for logarithmic functions ω, the above estimates were obtained in [6] and [14], respectively.

1.4. Organization of the paper

Section 2 is devoted to the radial divergence problem. In Section 3, we prove Theorem 1.2 and we show that estimate (2) is sharp, up to a multiplicative constant. Applications of Theorem 1.2 are presented in Section 4.

2. Radial divergence

Proposition 1.1(i) is a known fact. Indeed, if $I(0+) < \infty$ and $f \in \mathcal{B}^\omega(B_d)$, then $|\mathcal{R}f(z)|^2(1-|z|)$ is a Carleson measure, hence, $f \in \mathrm{BMOA}(B_d)$. In particular, f has radial limits σ_d-a.e.

2.1. Proof of Proposition 1.1(ii) for $d = 1$

Put

$$f(z) = \sum_{k=0}^{\infty} \omega\big(2^{-k}\big) z^{2^k}, \quad z \in B_1.$$

Standard arguments guarantee that $f \in \mathcal{B}^\omega(B_1)$. For example, let $t \in (0,1]$ and let $\tau = \frac{1}{t} \geqslant 1$. Observe that