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Characterizations of monotone $\mathcal{O}$-regularly varying functions by means of indefinite eigenvalue problems and HELP type inequalities

Andreas Fleige

Baroper Schulstrasse 27 A, 44225 Dortmund, Germany

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ABSTRACT

Connections between different areas of mathematical analysis are obtained: regular variation, indefinite operator theory and HELP type inequalities. Using a result by Parfyonov, the nondecreasing and so-called positively increasing functions induce precisely those indefinite Kreĭn–Feller eigenvalue problems such that the eigenfunctions have the Riesz basis property. Furthermore, these functions induce precisely those Lebesgue–Stieltjes measures such that the associated HELP type inequalities are valid. The so-called dual Kreĭn–Feller eigenvalue problems allow a similar characterization of the class of nondecreasing $\mathcal{O}$-regularly varying functions.

1. Introduction

It is well known that the classes of regularly and of $\mathcal{O}$-regularly varying functions and its various modifications play an important role in probability theory; see e.g. [4,24]. However, it is less popular that some of these classes also appear in indefinite operator theory. In particular, the so-called positively increasing functions (see e.g. [10,5]) appear in the recent papers [17,7], both dealing with indefinite Sturm–Liouville eigenvalue problems of the type $-u'' = \lambda ru$ where $xr(x) > 0$. Also a connection to HELP inequalities already follows from e.g. [2,3] (formulated in a different terminology) and [17,7]. However, these papers are all restricted to positively increasing functions m which are absolutely continuous, i.e. of the form $m(x) = \int_0^x r\,dt$.

For the general (i.e. non-absolutely continuous) case the present paper presents characterizations of the class of monotone $\mathcal{O}$-regularly varying (ORV) functions and of the class of monotone positively increasing (PI) functions by means of the following properties from formally different areas of mathematical analysis:

(i) the so-called Riesz basis property of an indefinite eigenvalue problem,
(ii) the non-singularity of the critical point infinity of the associated J-selfadjoint, J-nonnegative operator in a Kreĭn space,
(iii) the validity of a regular HELP type inequality.

The common platform for these characterizations is given by a signed Lebesgue–Stieltjes measure induced by a function m and the associated so-called indefinite Kreĭn–Feller differential operator studied in detail in [14]. Kreĭn–Feller operators can be regarded as an indefinite generalization of the concept of strings introduced in the fifties by Kreĭn and at the same time

E-mail address: andreas-fleige@versanet.de.

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by Feller; cf. [16,13,11]. Two different types of results are obtained in the paper: one for the function m itself characterizing the property PI and the other for its generalized inverse characterizing the property ORV by means of the "dual" Kreĭn–Feller operator or eigenvalue problem ("dual" in the sense of "dual strings"; cf. [16,11]). To this end a number of known results are collected and interpreted in a new framework.

In order to indicate the characterization of the property PI by (i) take a positive, nondecreasing and unbounded function f on $[a,\infty)$, $a>0$. Put $b:=1/a$ and $m_f(x):=1/f(1/x)$ for $x\in(0,b]$. This nondecreasing function induces a string on $[0,b]$. However, if we consider the even extension of m_f to $[-b,b]$ then m_f induces a signed Lebesgue–Stieltjes measure and hence, an indefinite Kreĭn–Feller differential expression $D_{m_f}D_xu$. Here, D_{m_f} is the Radon–Nikodym derivative with respect to the signed measure, again denoted by m_f. Note that $D_{m_f}D_xu$ coincides with the Sturm–Liouville expression u''/r if $m_f(x)=\int_0^x r\,dt$, $r\neq 0$. Also in general, the indefinite Kreĭn–Feller eigenvalue problem

$$-D_{m_f}D_xu=\lambda u,\qquad u(-b)=u(b)=0 \tag{1.1}$$

has a countable number of real and simple eigenvalues accumulating only at $+\infty$ and $-\infty$; cf. [14]. Let $\|m_f\|$ denote the total variation of m_f. In [14] and [21] the question was discussed whether the eigenfunctions form a Riesz basis in the Hilbert space $L^2_{\|m_f\|}$, i.e. an orthonormal basis with respect to some equivalent inner product; cf. [15]. This is property (i) and Parfyonov [21] proved that it is satisfied if and only if there is a constant $\omega\in(0,1)$ such that $m_f([0,\omega x))\leqslant\frac{1}{2}m_f([0,x))$ for all $x\in(0,b)$. It is not difficult to see that this is equivalent to the existence of a number $\omega\in(0,1)$ such that

$$\limsup_{x\searrow 0}\frac{m_f(\omega x)}{m_f(x)}<1.$$

This means by definition that m_f is positively increasing at 0 or, equivalently, that f is positively increasing at ∞.

For a characterization of the property ORV by (i) an observation by Djurčić, Nikolić and Torgašev is used. In [9] it was shown that f is $\mathcal{O}$-regularly varying at ∞ (i.e. $\limsup_{x\searrow\infty}f(\omega x)/f(x)<\infty$ for all $\omega>0$) if and only if the generalized inverse $f^{\leftarrow}(y)(=\inf\{x\geqslant a\mid f(x)>y\})$ is positively increasing at ∞. This is now equivalent to the Riesz basis property associated with the function $m_{f^{\leftarrow}}(y)=1/f^{\leftarrow}(1/y)$. It turns out that the corresponding indefinite Kreĭn–Feller eigenvalue problem is the dual problem to (1.1).

The first intention of the present paper is a detailed presentation of the above conclusions (so far ignoring some restrictions). Furthermore, the equivalence of (i) and (ii), obtained from a general result by Ćurgus and Najman [8], establishes a connection to the theory of definitizable operators in Kreĭn spaces; cf. [18,6].

In view of (iii), a criterion for (ii) by Ćurgus [6] is used in order to obtain a characterization of the properties PI and ORV by the inequality

$$\left(\int_0^b |D_{m_f}w|^2\,dm_f\right)^2\leqslant K\left(\int_0^b |w|^2\,dx\right)\left(\int_0^b |D_xD_{m_f}w|^2\,dx\right) \tag{1.2}$$

for all suitable functions w satisfying $(D_{m_f}w)(b)=0$. In case of an absolutely continuous function m_f (1.2) is a regular HELP inequality introduced e.g. in [2,12]. For this case the implication (i) $\Rightarrow$ (iii) was first observed by Volkmer in [25] and later extended to equivalence in [3]. If m_f is discrete then (1.2) is a discrete inequality and (1.1) a difference equation system. Inequality (1.2) in general seems to be new and only related to M. Langer's abstract approach in [19] where the inequality is characterized by associated boundary triplets and Titchmarsh–Weyl functions. Now, the property PI appears as an explicit (necessary and sufficient) condition on m_f for the validity of the inequality.

2. $\mathcal{O}$-regularly varying functions and positively increasing functions

In this section a number of definitions and known or basic results are collected.

2.1. Definitions and elementary properties

Let f be a positive, nondecreasing and unbounded function on $[a,\infty)$ with $a>0$. Furthermore, let m be a nondecreasing function on $[0,b]$ with $b>0$ such that

$$m(x)>0\quad\bigl(x\in(0,b]\bigr),\qquad 0=m(0)=\lim_{x\searrow 0}m(x). \tag{2.1}$$

Adapting the notations and definitions from e.g. [4,10,9,5,23], for $\omega>0$ put

$$f_{*,\infty}(\omega)=\liminf_{x\to\infty}\frac{f(\omega x)}{f(x)},\qquad f^{*,\infty}(\omega)=\limsup_{x\to\infty}\frac{f(\omega x)}{f(x)},$$
$$m_{*,0}(\omega)=\liminf_{x\searrow 0}\frac{m(\omega x)}{m(x)},\qquad m^{*,0}(\omega)=\limsup_{x\searrow 0}\frac{m(\omega x)}{m(x)}.$$