J. Math. Anal. Appl. 412 (2014) 401–415

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Omega theorems related to the general Euler totient function ☆

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ARTICLE INFO

Article history:
Received 25 June 2013
Available online 21 October 2013
Submitted by B.C. Berndt

Keywords:
Omega estimates
Generalized Euler totient function
Riemann zeta function
Dirichlet L-functions
Newforms L-functions

ABSTRACT

We prove an omega estimate related to the general Euler totient function associated to a polynomial Euler product satisfying some natural analytic properties. For convenience, we work with a set of L-functions similar to the Selberg class, but in principle our results can be proved in a still more general setup. In a recent paper the authors treated a special case of Dirichlet L-functions with real characters. Greater generality of the present paper invites new technical difficulties. Effectiveness of the main theorem is illustrated by corollaries concerning Euler totient functions associated to the shifted Riemann zeta function, shifted Dirichlet L-functions and shifted L-functions of modular forms. Results are either of the same quality as the best known estimates or are entirely new.

1. Introduction

Following [3] we define the general Euler totient function associated to a polynomial Euler product

$$F(s)=\prod_p F_p(s)=\prod_p\prod_{j=1}^{d}\left(1-\frac{\alpha_j(p)}{p^s}\right)^{-1},$$

where p runs over primes and $|\alpha_j(p)|\leqslant 1$ for all p and $1\leqslant j\leqslant d$ as follows

$$\varphi(n,F)=n\prod_{p|n}F_p(1)^{-1}. \tag{1.1}$$

We assume here that d is chosen as small as possible, i.e. that there exists at least one prime number p_0 such that

$$\prod_{j=1}^{d}\alpha_j(p_0)\neq 0.$$

Then d is called the *Euler degree* of F. Of course every polynomial Euler product is also a Dirichlet series

☆ Both authors were supported in part by the grant no. N N201 6059400 from the National Science Centre (Poland).
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0022-247X/$ – see front matter
http://dx.doi.org/10.1016/j.jmaa.2013.10.050

$$F(s)=\sum_{n=1}^{\infty}\frac{a_F(n)}{n^s}$$

which is absolutely convergent for $\sigma=\Re(s)>1$.

Let

$$\gamma(p)=p\left(1-\frac{1}{F_p(1)}\right) \tag{1.2}$$

and

$$C(F)=\frac{1}{2}\prod_p\left(1-\frac{\gamma(p)}{p^2}\right). \tag{1.3}$$

It was proved in [3, Theorem 1.1] that for a polynomial Euler product F of degree d and $x\geqslant 1$ we have

$$\sum_{n\leqslant x}\varphi(n,F)=C(F)x^2+O\big(x\big(\log(2x)\big)^d\big).$$

Let us denote the corresponding error term by

$$E(x,F)=\sum_{n\leqslant x}\varphi(n,F)-C(F)x^2. \tag{1.4}$$

As in [3] let $\mathcal{S}_0$ denote the set of all polynomial Euler products $F(s)$ belonging to the Selberg class $\mathcal{S}$ such that $F(s)\neq 0$ for

$$\sigma>1-\frac{c_0(F)}{\log(|t|+10)}\quad(s=\sigma+it,\ -\infty<t<\infty),$$

where $c_0(F)$ denotes a positive constant depending on F. We refer to [2,4,8,9] for basic definitions and results on the Selberg class. Let us remark that most probably $\mathcal{S}=\mathcal{S}_0$ but it has not been proven yet. It is convenient to introduce a more general class of L-functions as follows. We say that $F\in\tilde{\mathcal{S}}_0$ if there exists $F^*\in\mathcal{S}_0$ such that F and F^* differ by a finite number of local factors, i.e. there exists a finite set of primes T such that

$$\frac{F(s)}{F^*(s)}=\prod_{p\in T}\prod_{j=1}^{\partial_p}\left(1-\frac{\beta_j(p)}{p^s}\right)^{\epsilon_j(p)},$$

where $|\beta_j(p)|\leqslant 1$ and $\epsilon_j(p)\in\{-1,1\}$ for all $p\in T$ and $1\leqslant j\leqslant\partial_p$.

It is known that for $F\in\mathcal{S}_0$ we have

$$E(x,F)=\Omega(x) \tag{1.5}$$

as $x\to\infty$ (see [3, Corollary 1.4]). The principal goal of the present paper is to improve on this estimate. To formulate our main result we need the following auxiliary notation. For every prime number p and $\epsilon=\pm 1$ we put

$$\xi_p(F,\epsilon)=\arg\big(-\epsilon\gamma(p)\big)\quad\big(-\pi<\xi_p(F,\epsilon)\leqslant\pi\big) \tag{1.6}$$

and for every positive integer k let

$$\Psi_k(x,F,\epsilon)=\sum_{\substack{p\leqslant x\\ |\xi_p(F,\epsilon)|\leqslant\pi/2\\ p\equiv\epsilon\ (\mathrm{mod}\ k)}}\frac{|a_F(p)|}{p}\cos\xi_p(F,\epsilon). \tag{1.7}$$

Theorem 1.1. *Let $F(s)$ be a polynomial Euler product such that $F(s+i\lambda)\in\tilde{\mathcal{S}}_0$ for certain real λ. Suppose that $C(F)\neq 0$ (see (1.3)). Then, there exists a positive constant C which may depend on F such that for all integers $k>2$ and arbitrary $\epsilon=\pm 1$ we have*

$$E(x,F)=\Omega\big(x\exp\big(\Psi_k\big(C\varphi(k)\sqrt{\log x},F,\epsilon\big)\big)\big). \tag{1.8}$$

Obviously $\Psi_k(x,F,\epsilon)\geqslant 0$ for all x and hence we reprove (1.5) for a slightly larger class of L-functions but under additional but mild assumption $C(F)\neq 0$. However, in many concrete cases we have $\Psi_k(x,F,\epsilon)\to\infty$ as $x\to\infty$ for suitably chosen k and ϵ. In such cases we get an improvement of (1.5), see Theorems 2.1–2.3 below.