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# Hadamard three-hyperballs type theorem and overconvergence of special monogenic simple series

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ABSTRACT

The classical Hadamard three-circles theorem (1896) gives a relation between the maximum absolute values of an analytic function on three concentric circles. More precisely, it asserts that if $f$ is an analytic function in the annulus $\{z \in \mathbb{C}\colon r_1 < |z| < r_2\}$, $0 < r_1 < r < r_2 < \infty$, and if $M(r_1)$, $M(r_2)$, and $M(r)$ are the maxima of $f$ on the three circles corresponding, respectively, to $r_1$, $r_2$, and $r$ then

$$\{M(r)\}^{\log \frac{r_2}{r_1}} \leqslant \{M(r_1)\}^{\log \frac{r_2}{r}} \{M(r_2)\}^{\log \frac{r}{r_1}}.$$

In this paper we introduce a Hadamard's three-hyperballs type theorem in the framework of Clifford analysis. As a concrete application, we obtain an overconvergence property of special monogenic simple series.



## 1. Introduction

Two main problems that arise in the study of function spaces can be broadly described as follows:

1. Does the space under consideration possess a basis?
2. If this is the case, how can any other basis of this space be characterized?

These topics are closely linked together, but can be largely treated independently of each other. Let us assume for a moment that these problems are answered in a positive way. If $E$ denotes a topological space and $\{x_n\}_{n\in\mathbb{N}}$ a basis in $E$, then each element $x \in E$ admits a (unique) decomposition of the form $\sum_{n=1}^{\infty} a_n(x)x_n$ whereby for each $n \in \mathbb{N}$, $a_n$ is a linear functional on $E$. For the purposes of approximation theory the choice of a suitable basis is very important. This work deals essentially with these two fundamental problems in the case the underlying function spaces admit a set of polynomials as a basis. Classical examples of such function spaces are the space of holomorphic functions in an open disk and the space of analytic functions on a closed disk. Of course, as the theory of holomorphic functions in the plane allows higher dimensional generalizations [6], analogous problems may be considered in the corresponding function spaces.

In the early thirties Whittaker [34–37] and Cannon [7–9] have introduced the theory of basic sets (bases) of polynomials of one complex variable. This theory has been successfully extended to the Clifford analysis case in [2] (cf. [3]). Holomorphic

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functions (of one complex variable) are now replaced by Clifford algebra-valued functions that are defined in open subsets of $\mathbb{R}^{m+1}$ and that are solutions of a Dirac-type equation; for historical reasons they are called monogenic functions. In order to obtain a good analogy with the theory of one complex variable, the results in [2,3] have been restricted to polynomials with axial symmetry (also know as special polynomials), for which a Cannon theorem on the effectiveness could be proved in closed hyperballs. It should be observed that it is expected that a similar theory on basic sets of polynomials might be possible for polynomial null-solutions of generalized Cauchy–Riemann or Dirac operators, satisfying more general symmetry conditions. This matter is already well-exposed in [2,3] and essential ideas therein.

The main purpose of the present work is to introduce a Hadamard's three-hyperballs type theorem in the $(m+1)$-dimensional Euclidean space within the Clifford analysis setting by making use of the above-mentioned theory of basis of polynomials [2,3], and to establish an overconvergence property of special monogenic simple series. To the best of our knowledge this is done here for the first time. Theorems of this type have become significantly more involved in higher dimensions, and in particular in the quaternionic and Clifford analysis settings. In a series of papers [13,15,22,23], the authors have investigated higher dimensional counterparts of the well-known Bohr theorem and Hadamard real part theorems on the majorant of a Taylor's series, as well as Bloch's theorem, in the context of quaternionic analysis. These results provide powerful additional motivation to study the asymptotic growth behavior of monogenic functions from a given space, and to explore classical problems of the theory of monogenic quasi-conformal mappings [14,21] (see also [20, Ch. 3]).

For the general terminology used in this paper the reader is referred to Wittaker's book [37] in the complex case, and the work done by Abul-Ez et al. [2,3] in the Clifford analysis setting.

## 2. Preliminaries

### 2.1. Basic notions of Clifford analysis

The present subsection collects some definitions and basic algebraic facts of a special Clifford algebra of signature $(0,m)$, which will be needed throughout the text.

Let $\{e_1, e_2, \dots, e_m\}$ be an orthonormal basis of the Euclidean vector space $\mathbb{R}^m$ with a product according to the multiplication rules:

$$e_i e_j + e_j e_i = -2\delta_{i,j} \quad (i, j = 1, \dots, m),$$

where $\delta_{i,j}$ is the Kronecker symbol. This noncommutative product generates the $2^m$-dimensional Clifford algebra $Cl_{0,m}$ over $\mathbb{R}$, and the set $\{e_A\colon A \subseteq \{1, \dots, m\}\}$ with

$$e_A = e_{h_1} e_{h_2} \cdots e_{h_r}, \quad 1 \leqslant h_1 \leqslant \cdots \leqslant h_m,\ e_\phi = e_0 = 1,$$

forms a basis of $Cl_{0,m}$. The real vector space $\mathbb{R}^{m+1}$ will be embedded in $Cl_{0,m}$ by identifying the element $(x_0, x_1, \dots, x_m) \in \mathbb{R}^{m+1}$ with the algebra's element

$$x := x_0 + \mathbf{x} \in \mathcal{A}_m := \operatorname{span}_{\mathbb{R}}\{1, e_1, \dots, e_m\} \subset Cl_{0,m}.$$

The elements of $\mathcal{A}$ are usually called paravectors, and $x_0 := \operatorname{Sc}(x)$ and $e_1 x_1 + \cdots + e_m x_m := \mathbf{x}$ are the so-called scalar and vector parts of $x$. The conjugate of $x$ is $\bar{x} = x_0 - \mathbf{x}$, and the norm $|x|$ of $x$ is defined by

$$|x|^2 = x\bar{x} = \bar{x}x = x_0^2 + x_1^2 + \cdots + x_m^2.$$

As $Cl_{0,m}$ is isomorphic to $\mathbb{R}^{2^m}$ we may provide it with the $\mathbb{R}^{2^m}$-norm $|a|$, and one easily sees that for any $a, b \in Cl_{0,m}$, $|ab| \leqslant 2^{\frac{m}{2}} |a||b|$, where $a = \sum_{A \subseteq M} a_A e_A$ and $M$ stands for $\{1, 2, \dots, m\}$.

We consider $Cl_{0,m}$-valued functions defined in some open subset $\Omega$ of $\mathbb{R}^{m+1}$, i.e. functions of the form $f(x) := \sum_A f_A(x) e_A$, where $f_A(x)$ are scalar-valued functions defined in $\Omega$. Properties (like integrability, continuity or differentiability) that are ascribed to $f$ have to be fulfilled by all components $f_A$. In the sequel, we will make use of the generalized Cauchy–Riemann operator

$$D := \frac{\partial}{\partial x_0} + \sum_{i=1}^{m} e_i \frac{\partial}{\partial x_i}.$$

Suggested by the case $m = 1$, call a $Cl_{0,m}$-valued function $f$ left- (resp. right) *monogenic* in $\Omega$ if $Df = 0$ (resp. $fD = 0$) in $\Omega$. The interested reader is referred to [6] for more details.

Recent studies have shown that the construction of $\mathcal{A}_m$-valued monogenic functions as functions of a paravector variable is very useful, particularly if we study series expansions of $Cl_{0,m}$-valued functions in terms of special polynomial bases defined in $\mathbb{R}^{m+1}$. In this case we have

$$f : \Omega \subset \mathbb{R}^{m+1} \to \mathcal{A}_m, \quad f(x_0, \mathbf{x}) = f_0(x_0, \mathbf{x}) + \sum_{i=1}^{m} e_i f_i(x_0, \mathbf{x}),$$