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L^p convergence rates of solutions for n-dimensional quasilinear damped wave equation

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ABSTRACT

We realize the asymptotic profile for the solutions to the n-dimensional quasilinear damped wave equation in the divergence form is the solution of the corresponding linear parabolic equation with special initial data, and we further show the $L^p(1 \leqslant p \leqslant \infty)$ (in particular, $p=1$) convergence rates to this linear asymptotic profile.

1. Introduction

This paper is concerned with the diffusion phenomenon for the Cauchy problem of quasilinear damped wave equation in the divergence form

$$u_{tt} - \partial_i\big(f^i(\partial_i u)\big) + au_t = 0, \quad (x,t) \in \mathbb{R}^n \times (0,\infty), \tag{1.1}$$

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \tag{1.2}$$

where $\partial_i := \frac{\partial}{\partial x_i}$ and the summation convention over repeated indices is adopted in (1.1). Here a is a positive constant and $f = (f^1, f^2, \dots, f^n)$ is smooth with $f'(0) > 0$.

Matsumura [9] proved the global existence and asymptotics of the solutions for the Cauchy problem of the following second-order quasilinear hyperbolic equations with the first-order dissipation

$$L(u) = u_{tt} - \sum_{i,j=1}^{n} a_{ij}(x,t,Du)u_{ij} + \alpha u_t + b(Du) = 0,$$

where $x \in \mathbb{R}^n$, $t \geqslant 0$, $\alpha > 0$, $u_i = \frac{\partial u}{\partial x_i}$, $u_t = \frac{\partial u}{\partial t}$, and

$$Du = (u, u_t, u_1, u_2, \dots, u_n).$$

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Here the coefficients a_{ij} are smooth and satisfy

$$\sum_{i,j} a_{ij}(x,t,y)\xi_i\xi_j \geqslant a(y)\sum_i \xi_i^2, \quad a(0) > 0$$

for $x \in \mathbb{R}^n$, $t \in \mathbb{R}^1$, $y \in \mathbb{R}^{n+2}$, $\xi \in \mathbb{R}^n$.

For $n = 1$, under the assumptions that $(u_0(x), u_1(x))$ lies in $(H^3 \times H^2)(\mathbb{R}) \cap (L^1 \times L^1)(\mathbb{R})$ and is small in $(H^3 \times H^2)(\mathbb{R})$, Nishihara [14] showed the Cauchy problem (1.1), (1.2) admits a unique global smooth solution $u(x,t)$ which satisfies

$$\big\|\big(u-\psi, (u-\psi)_x, (u-\psi)_t\big)(t)\big\|_{L^\infty} = O(1)\big(t^{-1}, t^{-3/2}, t^{-2}\big). \tag{1.3}$$

Here $\psi(x,t)$ is the unique global solution of the Cauchy problem for the corresponding parabolic equation

$$a\psi_t - {f^1}'(0)\psi_{xx} = 0, \tag{1.4}$$

$$\psi(x,0) = u_0(x) + \frac{1}{a}u_1(x). \tag{1.5}$$

Under some other smallness assumptions on the initial data $(u_0(x), u_1(x))$, if $\int_{-\infty}^{\infty}(u_0(x) + \frac{1}{a}u_1(x))\,dx = 0$, $(u_0(x), u_1(x)) \in (H^5 \times H^4)(\mathbb{R}) \cap (L^1 \times L^1)(\mathbb{R})$, H.-J. Zhao [24] has proved that $\psi(x,t)$ is still an asymptotic profile of $u(x,t)$ and satisfies

$$\big\|\big(u-\psi, (u-\psi)_x, (u-\psi)_t\big)(t)\big\|_{L^\infty} = O(1)\big(t^{-\frac{3}{2}}, t^{-2}, t^{-\frac{5}{2}}\big).$$

Recently, for the case of $\int_{-\infty}^{\infty}(u_0(x) + \frac{1}{a}u_1(x))\,dx = 0$, by suitably choosing the initial data of the parabolic equation, S.-F. Geng [2] showed the solution Ψ of the corresponding parabolic equation served as the new asymptotic profile of the solutions for the problem (1.1), (1.2) satisfies

$$\big\|\big(u-\Psi, (u-\Psi)_x, (u-\Psi)_t\big)(t)\big\|_{L^\infty} = O(1)\big(t^{-2}, t^{-\frac{5}{2}}, t^{-3}\big),$$

which is better than that of ψ defined by (1.4), (1.5). Here $\Psi(x,t)$ satisfies the following Cauchy problem

$$a\Psi_t - {f^1}'(0)\Psi_{xx} = 0, \tag{1.6}$$

$$\Psi(x,0) = u_0(x) + \frac{1}{a}u_1(x) + \int_0^\infty F^1(u_x)_x\,d\tau, \tag{1.7}$$

where

$$F^1(u_x) = f^1(u_x) - f^1(0) - {f^1}'(0)u_x. \tag{1.8}$$

The main purpose of our present paper is to extend the L^∞-estimate by Yang–Milani in [22,23] to the L^p-estimate ($1 \leqslant p \leqslant \infty$) (note that the L^1-estimate is also obtained in [11], but the results there are only in linear problem).

As in [14], we define the profile $\phi(x,t)$ of the solutions $u(x,t)$ for the problem (1.1), (1.2) as follows:

$$a\phi_t - {f^i}'(0)\partial_i^2\phi = 0, \tag{1.9}$$

$$\phi(x,0) = u_0(x) + \frac{1}{a}u_1(x). \tag{1.10}$$

Then, we can prove that the solution u to the Cauchy problem (1.1), (1.2) converges to the solution ϕ of the problem (1.9), (1.10) for the linear equation and satisfies the convergence rate (see (1.16) below).

These results indicate that, as $t \to \infty$, Eq. (1.1) has a parabolic structure. Such an observation was originally observed by Hsiao and Liu [4], for the system of hyperbolic conservation laws with damping

$$v_t - u_x = 0, \tag{1.11}$$

$$u_t + p(v)_x = -\alpha u, \tag{1.12}$$

with smooth initial data $u(x,0) = u_0(x)$, $v(x,0) = v_0(x)$, that are asymptotically constant, that is,

$$\big(u_0(x), v_0(x)\big) \to (u_\pm, v_\pm), \quad \text{as } x \to \infty.$$

In (1.11), (1.12), it assumed that $p(v) > 0$, $p'(v) < 0$, for $v > 0$, and $v_0(x), v_\pm > 0$. Hsiao and Liu showed that the solutions to (1.11), (1.12) time asymptotically behave like those governed by the Darcy's law in L^2 and L^∞ norms. That is, as t tends to ∞, the smooth solution $(v(x,t), u(x,t))$ which is away from vacuum approaches to the solution $(\bar{v}(x,t), \bar{u}(x,t))$ governed by the following system with the same initial data: