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Weak sequential convergence in $L^1(\mu, X)$ and an exact version of Fatou's lemma ☆

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ABSTRACT

The class of nonatomic finite measure spaces with the saturation property, as developed in Maharam (1942) and Hoover–Keisler (1984), is characterized by the Fatou (and Lebesgue) property of a well-dominated sequence of multifunctions taking values in a Banach space. With multifunctions reduced to functions, this Fatou characterization also extends to a variant of the closure property found in optimal control theory. The results are developed through a considered overview of the relevant literature on the exact and approximate Fatou lemma phrased in terms of Bochner integration.

Contents

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1. Introduction

Fatou's lemma for real-valued functions is a basic result in the theory of Lebesgue integration, and motivated by issues arising in mathematical economics in the late sixties, it was generalized in [30] to functions taking values in a finite-dimensional Euclidean space. The subject has since grown substantially, and a considerable literature pertaining to both functions and multifunctions taking values in infinite-dimensional separable Banach spaces has accumulated; see [14, Section 12] for a survey up till 2002, and [24] and their references for more recent work. An important benchmark for this work has been the celebrated theorem of Lyapunov, and in particular, his example of its failure for infinite-dimensional spaces; see [8, Chapter IX]. As a consequence, Fatou's lemma has been presented in an *exact* or an *approximate* form depending on whether the context is finite- or infinite-dimensional, and the underlying abstract finite measure space, nonatomic or purely atomic.

In work done in the nineties, exact versions of Fatou's lemma for multifunctions taking values in a separable Banach space were proved in [31,32], but under the specific condition that the underlying measure space belongs to a class discovered by Loeb [23]. This restriction to Loeb spaces has proved fruitful in the isolation of the *saturation* property due to Hoover and Keisler [16] that goes back to Maharam's classification [25], as well as to a development of a substantial theory based on that property; see [10].[1] Saturated measure spaces have now been characterized by properties of integration and distribution of multifunctions, and more recently, by the Lyapunov property; see [21] and its references.[2] Since Fatou's lemma holds trivially for a measure space constituted by a finite number of atoms, and saturated measure spaces are nonatomic, it would be naive to ask whether the saturation and Fatou properties are equivalent; but as shown in this paper, a complete characterization can indeed be furnished under the restriction to nonatomic finite measure spaces. As the proofs of the exact versions of Fatou's lemma presented below make clear, they result from a combination of the approximate versions with theorems on integrals of multifunctions in [27, Theorem 2] or [33, Proposition 1].[3] And in the case of functions, this equivalence can also be connected to a variant of the closure property identified in control systems by Cesari and his followers; see [6] for example.

The principal results of the paper are Theorems 3.7 and 4.9, and after a section on the necessary preliminaries, they are presented in Sections 3 and 4: the first concerning functions, and the second, multifunctions. Even though the question investigated here has not been formally posed before, the affirmative answer obtained to it is of considerable theoretical and applied interest.

2. Preliminaries

Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space and $(X, \|\cdot\|)$ be a Banach space with its dual X^*. Denote by $L^1(\mu, X)$ the space of (the equivalence classes of) X-valued Bochner integrable functions defined on Ω, normed by $\|f\|_1 = \int \|f\|\, d\mu$, $f \in L^1(\mu, X)$. The dual space of $L^1(\mu, X)$ is given by $L^\infty(\mu, X^*_{w^*})$, where $X^*_{w^*}$ denotes the dual space X^* endowed with the weak* topology (see [17, Corollary to Theorem VII.4.7]) and the dual system is given by $\langle f^*, f\rangle = \int \langle f^*(t), f(t)\rangle\, d\mu$ with $f \in L^1(\mu, X)$ and $f^* \in L^\infty(\mu, X^*_{w^*})$. Since any element of X^* is regarded as an element of $(L^1(\mu, X))^* = L^\infty(\mu, X^*_{w^*})$, it is easy to verify that the integration operator $f \mapsto \int f\, d\mu$ is continuous in the weak topologies for $L^1(\mu, X)$ and X.

The *strong upper limit* of a sequence $\{S_n\}$ of subsets in X is defined by

$$\operatorname{Ls} S_n = \Big\{x \in X \;\Big|\; \exists \{x_{n_i}\}\colon\ x = \lim_{i\to\infty} x_{n_i},\ x_{n_i} \in S_{n_i}\ \forall i \in \mathbb{N}\Big\},$$

where $\{x_{n_i}\}$ denotes a subsequence of $\{x_n\} \subset X$. We denote by $w\text{-}\lim_n x_n$ the weak limit point of a sequence $\{x_n\}$ in X. The *weak upper limit* of $\{S_n\}$ of subsets in X is defined by

$$w\text{-}\operatorname{Ls} S_n = \Big\{x \in X \;\Big|\; \exists \{x_{n_i}\}\colon\ x = w\text{-}\lim_{i\to\infty} x_{n_i},\ x_{n_i} \in S_{n_i}\ \forall i \in \mathbb{N}\Big\}.$$

The *weak lower limit* of $\{S_n\}$ is defined by

$$w\text{-}\operatorname{Li} S_n = \Big\{x \in X \mid x = w\text{-}\lim_{n\to\infty} x_n,\ x_n \in S_n\ \forall n \in \mathbb{N}\Big\}.$$

A set S is called the *weak limit* of $\{S_n\}$ if $w\text{-}\operatorname{Ls} S_n = w\text{-}\operatorname{Li} S_n = S$ and denoted by $S = w\text{-}\operatorname{Lim} S_n$.

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[1] See the paragraph immediately following Definition 3.4 below for this property. In his monumental treatise on *Measure Theory*, Fremlin [11,12] uses the word *saturation* always followed by an *of*, as in *saturation of* a Boolean algebra, or of a pre- or partially ordered set, or of a forcing notion, or of a supported relation, or of a topological space; see the index to the word in [11]. He never uses it to refer to a property of a measure space as in this paper. Note also that the *saturation principle* is well-known to workers in nonstandard analysis; see [10, p. 24]. In this paper we never use the word as in any of these phrases, or on its own, but only as an adjective qualifying *property*. Thus there is little possibility of any confusion.

[2] See Definition 3.4 below, and also footnote 1 above. Again, Fremlin [12, Remark to 316C] uses the word *saturated* only in a qualified way (κ-saturated and ω_1-saturated) for an ideal of a Dedekind complete Boolean algebra, and never as an adjective for a property for a measure space. And so, there is again little possibility of any confusion with respect to earlier usages of this word.

[3] We are indebted to our anonymous referee for this insight.