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Generalized bi-circular projections on certain Hardy spaces

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ABSTRACT

We characterize generalized bi-circular projections on the Hardy space of the torus and show that they can be represented as the average of the identity and an isometric reflection. We also show that the average of two distinct isometries on the Hardy space of the disk is a projection if and only if it is a generalized bi-circular projection.

1. Introduction

In many ways the projections on a given Banach space give us an insight into the geometric properties of the space. For this reason, characterizations of various types of projections have received considerable interest in the past 30 years. Bi-contractive projections, i.e those bounded linear operators P on a Banach space X with property that $\|P\| \leq 1$ and $\|I - P\| \leq 1$, have been the focus of several research articles. For more information, we refer the reader to [2,8,9,11]. One class of projections which have been shown to be bi-contractive and has received considerable attention in recent years are the generalized bi-circular projections (GBP). We recall the following definition from [7].

Definition 1.1. A projection P on a Banach space X is a GBP if $P + \lambda(I - P)$ is an isometry of X for some choice of $\lambda(\neq 1)$ with $|\lambda| = 1$.

In the first section of the paper we characterize the GBPs on the H^p spaces of the torus using a result of Lal and Merrill. We find that they can be represented as the average of the identity and an isometric reflection. In the second part of the paper we consider the classical H^p spaces of the disk and determine when the average of two isometries is a projection. We find indeed this only happens when the projection in question is a GBP. This coincides with results found by Botelho for $C(\Omega)$ [4] and Botelho and Jamison for $C(\Omega, E)$ [3].

2. Bi-circular projections on H^p spaces of the Torus

Let $A(\mathbb{T}^2)$ denote the algebra of continuous, complex valued functions on the torus $\mathbb{T}^2 = \{(z, w) : |z| = |w| = 1\}$ which are uniform limits of polynomials in $z^n w^m$ where $(m, n) \in \mathcal{S} = \{(m, n) : n > 0\} \cup \{(m, 0) : m \geq 0\}$. Let dm denote the normalized Haar measure on $\mathbb{T}^2$ and $\mathbf{H}^p$ denote the Banach space consisting of the closure of $A(\mathbb{T}^2)$ in $L^p(dm)$ (norm closure for $1 \leq p < \infty$, w^*-closure for $p = \infty$). In this paper we characterize the generalized bicircular projections on $\mathbf{H}^p(\mathbb{T}^2)$. Berkson and Porta note that the study of this space is more involved due to the asymmetrical nature of the independent variables (see [1]). We recall a result of Lal and Merrill [10].

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Theorem 2.1. *A linear operator T of $H^p(\mathbb{T}^2)$ onto $H^p(\mathbb{T}^2)$ ($1 \le p < \infty$, $p \neq 2$) is an isometry if and only if*

$$(Tf)(z, w) = \alpha(\tau'(z))^{1/p} f(\tau(z), w\sigma(z))$$

for all $f \in H^p$ where $|z| = |w| = 1$, α is a complex constant of modulus 1, τ is a conformal map of the unit disk onto itself, and σ is a unimodular measurable function on the circle.

It follows from the definition of generalized bicircular projection that the associated isometry must indeed be a surjective isometry. We use this to find the generalized bi-circular projections on $H^p(\mathbb{T}^2)$.

Theorem 2.2. *P is a generalized bi-circular projection on $H^p(\mathbb{T}^2)$ if and only if P is trivial or*

$$Pf(z, w) = \frac{1}{2}(\pm(\tau'(z))^{1/p} f(\tau(z), w\sigma(z)) + f(z, w))$$

where τ is a conformal map of the unit disk onto itself such that $\tau(\tau(z)) = z$ and σ is a unimodular measurable function on the circle such that $\sigma(z)\sigma(\tau(z)) = 1$.

Proof. P is a generalized bi-circular projection on $H^p(\mathbb{T}^2)$ if and only if $P = \frac{T-\lambda I}{1-\lambda}$ where T is a surjective isometry on $H^p(\mathbb{T}^2)$. $P^2 = P$ implies that for every $f \in H^p(\mathbb{T})$ and every $x, y \in \mathbb{T}^2$

$$\begin{aligned}&\alpha^2(\tau'(z))^{1/p}(\tau'(\tau(z)))^{1/p} f(\tau(\tau(z)), w\sigma(z)\sigma(\tau(z)))\\ &\quad -(\lambda+1)\alpha(\tau'(z))^{1/p} f(\tau(z), w\sigma(z)) + \lambda f(z, w) = 0.\end{aligned} \tag{1}$$

In particular, for $f(z, w) = 1$, (1) yields $\alpha^2(\tau'(\tau z))^{1/p} = (\lambda+1)\alpha(\tau'(z))^{1/p} - \lambda$. Then (1) reduces to

$$\begin{aligned}&(\lambda+1)\alpha(\tau'(z))^{1/p}[f(\tau(\tau(z)), w\sigma(z)\sigma(\tau(z))) - f(\tau(z), w\sigma(z))]\\ &\quad = \lambda[f(\tau(\tau(z)), w\sigma(z)\sigma(\tau(z))) - f(z, w)].\end{aligned} \tag{2}$$

Now consider $f(z, w) = z$ and $f(z, w) = z^2$. Then (2) reduces to

$$(\lambda+1)\alpha(\tau'(z))^{1/p}[\tau(\tau(z)) - \tau(z)] = \lambda[\tau(\tau(z)) - z] \tag{3}$$

and

$$(\lambda+1)\alpha(\tau'(z))^{1/p}[(\tau(\tau(z)))^2 - (\tau(z))^2] = \lambda[\tau(\tau(z))^2 - z^2], \tag{4}$$

respectively. By substituting (3) into (4) we get

$$(\tau(\tau(z)) - z)(\tau(z) - z) = 0.$$

Thus, $\forall z$, $\tau(\tau(z)) = z$ or $\tau(z) = z$. In either case $\tau(\tau(z)) = z$.

Note that by substituting $z = \tau(\tau(z))$ into (3) we get $(\lambda+1)(\tau'(z))^{1/p}[z - \tau(z)] = 0$. Therefore $\lambda = -1$ or $\forall z$, $\tau(z) = z$.

Now consider $f(z, w) = w$ and $f(z, w) = w^2$. Then (2) reduces to

$$(\lambda+1)\alpha(\tau'(z))^{1/p}[w\sigma(z)\sigma(\tau(z)) - w\sigma(z)] = \lambda[w\sigma(z)\sigma(\tau(z)) - w] \tag{5}$$

and

$$(\lambda+1)\alpha(\tau'(z))^{1/p}[(w\sigma(z)\sigma(\tau(z)))^2 - (w\sigma(z))^2] = \lambda[(w\sigma(z)\sigma(\tau(z)))^2 - w^2], \tag{6}$$

respectively.

Substituting (5) into (6) we get

$$(\sigma(z)\sigma(\tau(z)) - 1)(\sigma(z) - 1) = 0.$$

Thus, $\forall z$, $\sigma(z)\sigma(\tau(z)) = 1$ or $\sigma(z) = 1$. Note that if $\tau(z) = z$, then $\sigma(z) = \pm 1$. However, since we showed earlier that $\tau(\tau(z)) = z$ for all z, this implies $\sigma(z)\sigma(\tau(z)) = 1$ for all z. Then (1) can be reduced to

$$\alpha^2 f(z, w) - (\lambda+1)\alpha(\tau'(z))^{1/p} f(\tau(z), w\sigma(z)) + \lambda f(z, w) = 0. \tag{7}$$

There are two cases to consider.

(1) If for all z, $\tau(z) = z$ and $\sigma(z) = 1$, then the projections are $Pf(z, w) = 0$ and $Pf(z, w) = f(z, w)$.
(2) If $\tau(\tau(z)) = z$ and $\sigma(z)\sigma(\tau(z)) = 1$, then recall from earlier that $\lambda = -1$. Then (7) can be written as

$$\alpha^2 f(z, w) - f(z, w) = 0.$$

Thus, $\alpha = \pm 1$.

This yields the projections $P = \frac{1}{2}(\pm(\tau'(z))^{1/p} f(\tau(z), w\sigma(z)) + f(z, w))$. □