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Division problem for spatially periodic distributions

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ABSTRACT

We give a sufficient condition for the surjectivity of partial differential operators with constant coefficients on the space of distributions on $\mathbb{R}^{n+1}$ (here we think of there being n space directions and one time direction) that are periodic in the spatial directions and tempered in the time direction.

1. Introduction

An important milestone in the general theory of partial differential equations is the solution to the division problem: Let D be a nonzero partial differential operator with constant coefficients and T be a distribution; can one find a distribution S such that $DS = T$? That this is always possible was established by Ehrenpreis [3]. See also [1,2,4–7,10] for the solution of various avatars of the division problem for different spaces of distributions.

In this article, we study the division problem in spaces of distributions on $\mathbb{R}^{n+1}$ (where we think of there being n space directions and one time direction) that are periodic in the spatial directions. The study of such solution spaces arises naturally in control theory when one considers the so-called "spatially invariant systems"; see [8]. In the "behavioural approach" to control theory for such spatially invariant systems, a fundamental question is whether this class of distributions is an injective module over the ring of partial differential operators with constant coefficients; see for example [11]. In light of this, one can first ask what happens with the division problem. Thus besides being a purely mathematical question that fits into the classical theme mentioned in the previous paragraph, there is also a behavioural control theoretic motivation for studying the division problem for distributions that are periodic in the spatial directions. Upon taking Fourier transform with respect to the spatial variables, the problem amounts to the following.

Problem. For which $P(\tau, \boldsymbol{\xi}) \in \mathbb{C}[\tau, \xi_1, \dots, \xi_n]$ is $P\left(\frac{d}{dt}, i\boldsymbol{\xi}\right) : X \to X$ surjective, where

$$X = \{(T_{\boldsymbol{\xi}}) = (T_{\boldsymbol{\xi}})_{\boldsymbol{\xi}\in\mathbb{Z}^n} \in \mathcal{D}'(\mathbb{R})^{\mathbb{Z}^n} : \forall \varphi \in \mathcal{D}(\mathbb{R}),\ \exists k \in \mathbb{N} :\ \forall \boldsymbol{\xi} \in \mathbb{Z}^n,\ |\langle \varphi, T_{\boldsymbol{\xi}}\rangle| \le k \cdot (1 + |\boldsymbol{\xi}|)^k\}?$$

($|\cdot|$ denotes the 1-norm in $\mathbb{R}^n$.)

An obvious necessary condition is that

$$\text{for all } \boldsymbol{\xi} \in \mathbb{Z}^n, \qquad P\left(\frac{d}{dt}, i\boldsymbol{\xi}\right) \not\equiv 0.$$

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However, that this condition is not sufficient is demonstrated by considering the following example.

Example 1.1. Let $c = \sum_{j=1}^{\infty} \frac{1}{2^{j!}}$ (a "Liouville number"). With $p_k := \sum_{j=1}^{k} 2^{k!-j!}$, $q_k := 2^{k!}$,

$$\left|c - \frac{p_k}{q_k}\right| = \sum_{j=k+1}^{\infty} \frac{1}{2^{j!}} = \frac{1}{2^{(k+1)!}} + \frac{1}{2^{(k+2)!}} + \cdots < 2 \cdot \frac{1}{2^{(k+1)!}} \leq \left(\frac{1}{2^{k!}}\right)^k = \frac{1}{q_k^k}, \quad k \in \mathbb{N}.$$

Now take $P = \xi_1 + c\xi_2$. Then $(1)_{\xi \in \mathbb{Z}^2}$ does not belong to the range of P because

$$\left(\frac{1}{\xi_1 + c\xi_2}\right)_{\xi \in \mathbb{Z}^2} \notin X.$$

Indeed, otherwise there would exist an m such that

$$\frac{1}{|\xi_1 + c\xi_2|} \leq m(1 + |\xi_1| + |\xi_2|)^m \quad \text{for all } \xi_1, \xi_2 \in \mathbb{Z},$$

and in particular, with $\xi_1 = -p_k$, and $\xi_2 = q_k$, $k \in \mathbb{N}$,

$$q_k^{k-1} \leq \frac{1}{|\xi_1 + c\xi_2|} \leq m(1 + p_k + q_k)^m \leq m(q_k + q_k + q_k)^m = m3^m q_k^m,$$

a contradiction. $\square$

We consider a simpler situation and set

$$Y = \{(T_{\boldsymbol{\xi}}) \in \mathcal{S}'(\mathbb{R})^{\mathbb{Z}^n} : \forall \varphi \in \mathcal{S}(\mathbb{R}),\ \exists k \in \mathbb{N} : \forall \boldsymbol{\xi} \in \mathbb{Z}^n,\ |\langle \varphi, T_{\boldsymbol{\xi}} \rangle| \leq k \cdot (1 + |\boldsymbol{\xi}|)^k\}.$$

Our main result is the following:

Theorem 1.2. *Let $P(\tau, i\boldsymbol{\xi}) \in \mathbb{C}[\tau, \boldsymbol{\xi}] = \mathbb{C}[\boldsymbol{\xi}][\tau]$, and for $\boldsymbol{\xi} \in \mathbb{Z}^n$,*

$$P(\tau, i\boldsymbol{\xi}) = c_{\boldsymbol{\xi}} \cdot \prod_{j=1}^{m_{\boldsymbol{\xi}}} (\tau - \lambda_{j,\boldsymbol{\xi}}),$$

with $m_{\boldsymbol{\xi}} \in \mathbb{N}_0$, $c_{\boldsymbol{\xi}} \in \mathbb{C} \setminus \{0\}$, $\lambda_{1,\boldsymbol{\xi}}, \ldots, \lambda_{m_{\boldsymbol{\xi}},\boldsymbol{\xi}} \in \mathbb{C}$. (The roots $\lambda_{j,\boldsymbol{\xi}}$ are arbitrarily arranged.) Let

$$d_{\boldsymbol{\xi}} := \begin{cases} 1 & \text{if for all } j = 1, \ldots, m_{\boldsymbol{\xi}},\ \mathrm{Re}(\lambda_{j,\boldsymbol{\xi}}) = 0, \\ \min\limits_{j:\mathrm{Re}(\lambda_{j,\boldsymbol{\xi}}) \neq 0} |\mathrm{Re}(\lambda_{j,\boldsymbol{\xi}})| & \text{otherwise}. \end{cases}$$

If

(1) $(c_{\boldsymbol{\xi}}^{-1}) \in s'(\mathbb{Z}^n)$ *and*

(2) $(d_{\boldsymbol{\xi}}^{-1}) \in s'(\mathbb{Z}^n)$,

then $P\left(\frac{d}{dt}, i\boldsymbol{\xi}\right) : Y \to Y$ is surjective.

From Example 1.1, it follows that the first condition is not superfluous. Here is an example demonstrating that the second condition is not superfluous either.

Example 1.3. Take

$$P\left(\frac{d}{dt}, i\boldsymbol{\xi}\right) := \frac{d}{dt} + \xi_1 + c\xi_2$$

with the same c as in Example 1.1, and $T_{\boldsymbol{\xi}} := 1$, $\boldsymbol{\xi} \in \mathbb{Z}^2$. Then the constant $S_{\boldsymbol{\xi}} = 1/(\xi_1 + c\xi_2)$ is the only solution in $\mathcal{S}'(\mathbb{R})$ of the equation

$$P\left(\frac{d}{dt}, i\boldsymbol{\xi}\right) S_{\boldsymbol{\xi}} = T_{\boldsymbol{\xi}} = 1,$$

but the sequence

$$(S_{\boldsymbol{\xi}}) = \left(\frac{1}{\xi_1 + c\xi_2}\right)$$

does not belong to Y and hence $(1)_{\boldsymbol{\xi} \in \mathbb{Z}^2}$ is not contained in the range of P. $\square$