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Mathematical models of a hydraulic shock

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ABSTRACT

In the present paper, we derive macroscopic mathematical models of the pressure distribution field near the oil well during the hydraulic shock. To get these models we follow the scheme, suggested by J. Keller and R. Burridge. This scheme is based upon a rigorous homogenization of the exact mathematical model, describing on a microscopic level the joint motion of an elastic solid skeleton and a viscous fluid filling the pores.

0. Introduction

Hydraulic shock is a sharp rise of the pressure in some fluid-filled system like pipes, cracks and pores. This process in an oil well is a part of the hydraulic fracturing. There are some engineering models (formulas) to calculate the pressure in the pipe system during the hydraulic shock. But these models do not work for more complex systems, such as oil well. Existing mathematical models of the hydraulic shock in porous media [1,4,5] are nothing more than the same engineering models as for the pipe systems.

In the present paper we derive macroscopic mathematical models of the hydraulic shock, which are asymptotically close to the basic models of continuum mechanics. To do this, we follow a very natural idea of R. Burridge and J. Keller [2]: first of all, describe the physical process on the microscopic level by some physically correct mathematical model, and then, if there is a small parameter, find all limiting regimes (homogenized equations) as the small parameter goes to zero.

As a basic mathematical model describing the hydraulic shock on the microscopic level we consider a model of short-time isothermal processes in an incompressible medium [9,2,6–8], where the dimensionless displacement vector $\boldsymbol{w}$ of the continuum medium in the dimensionless variables

$$\boldsymbol{x}' = L\boldsymbol{x}, \qquad t' = \tau t, \qquad \boldsymbol{w}' = \frac{L^2}{g\tau^2}\boldsymbol{w}$$

satisfies the differential equation in the domain Ω for $t > 0$:

$$\nabla \cdot \boldsymbol{w} = 0, \tag{0.1}$$

$$\tilde{\varrho}\frac{\partial^2 \boldsymbol{w}}{\partial t^2} = \nabla \cdot \mathbb{P} + \tilde{\varrho}\boldsymbol{F}, \tag{0.2}$$

$$\mathbb{P} = \tilde{\chi}\bar{\alpha}_\mu \mathbb{D}\left(x, \frac{\partial \boldsymbol{w}}{\partial t}\right) + (1-\tilde{\chi})\bar{\alpha}_\lambda \mathbb{D}(x, \boldsymbol{w}) - p\,\mathbb{I}. \tag{0.3}$$

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Here

$$\tilde{\varrho} = \tilde{\chi}\varrho_f + (1-\tilde{\chi})\varrho_s,$$

$\tilde{\chi}(\boldsymbol{x})$ is a characteristic function of the pore space, $p(\boldsymbol{x}, t)$ is the pressure, ρ_f and ρ_s are the mean dimensionless densities of the fluid and rigid solid components respectively, scaled with the mean density of the water ρ_0, $\mathbb{D}(x, \boldsymbol{u})$ is a symmetric part of $\nabla\boldsymbol{u}$:

$$\mathbb{D}(x, \boldsymbol{u}) = \frac{1}{2}\left(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^*\right),$$

and $\mathbb{I}$ is the unit tensor.

Dimensionless criteria $\bar{\alpha}_\mu$ and $\bar{\alpha}_\lambda$ are defined by the formulas

$$\bar{\alpha}_\mu = \frac{2\mu\tau}{L^2\rho_0}, \qquad \bar{\alpha}_\lambda = \frac{2\lambda\tau^2}{L^2\rho_0},$$

where μ is the viscosity of fluid, λ is elastic Lamé's constant, τ is a characteristic time of the process and L is the characteristic size of the domain in consideration.

Eq. (0.2) is understood in a sense of distribution and contains Stokes equations in the liquid part, Lamé's equations in the solid skeleton and the continuity condition for the normal stresses on the common boundary "solid skeleton–pore space".

This mathematical model contains a natural small parameter ε, which is a characteristic size of pores l divided by the characteristic size L: $\varepsilon = l/L$.

Our aim is to derive all possible limiting regimes (the homogenized equations) as $\varepsilon \searrow 0$. Such an approximation significantly simplifies the original problem and at the same time preserves all of its main features. But even this approach is too difficult to be realized, and some additional simplifying assumptions are necessary. In terms of geometrical properties of the medium, it is most expedient to simplify the problem by postulating that the porous structure is periodic.

We impose the following constraints.

Assumption 1. (1) Let Y_s be the "solid part" of the unit cube $Y = (0, 1)^3 \subset \mathbb{R}^3$, and let the "liquid part" Y_f of Y be its open complement. We write $\gamma = \partial Y_f \cap \partial Y_s$ and assume that γ is a Lipschitz continuous surface.

(2) The domain E_f is a periodic repetition in $\mathbb{R}^3$ of the elementary cell $Y_f^\varepsilon = \varepsilon Y_f$ and the domain E_s is a periodic repetition in $\mathbb{R}^3$ of the elementary cell $Y_s^\varepsilon = \varepsilon Y_s$.

(3) The pore space $\Omega_f^\varepsilon \subset \Omega = \Omega \cap E_f$ is a periodic repetition in Ω of the elementary cell εY_f, and the solid skeleton $\Omega_s^\varepsilon \subset \Omega = \Omega \cap E_s$ is a periodic repetition in Ω of the elementary cell εY_s. The Lipschitz continuous boundary $\Gamma^\varepsilon = \partial\Omega_s^\varepsilon \cap \partial\Omega_f^\varepsilon$ is a periodic repetition in Ω of the boundary $\varepsilon\gamma$.

(4) $\Omega = \Omega_f^\varepsilon \cup \Gamma^\varepsilon \cup \Omega_s^\varepsilon$. The pore space Ω_f^ε and the solid skeleton Ω_s^ε are connected domains.

Under these assumptions

$$\tilde{\chi}(\boldsymbol{x}) = \chi^\varepsilon(\boldsymbol{x}) = \chi_0(\boldsymbol{x})\chi\left(\frac{\boldsymbol{x}}{\varepsilon}\right),$$

where $\chi_0(\boldsymbol{x})$ is a characteristic function of the domain Ω.

We suppose that the dimensionless parameters $\bar{\alpha}_\mu$ and $\bar{\alpha}_\lambda$ depend on the small parameter ε and the (finite or infinite) limits exist:

$$\lim_{\varepsilon\searrow 0}\bar{\alpha}_\mu(\varepsilon) = \mu_0, \qquad \lim_{\varepsilon\searrow 0}\frac{\bar{\alpha}_\mu}{\varepsilon^2} = \mu_1, \qquad \lim_{\varepsilon\searrow 0}\bar{\alpha}_\lambda(\varepsilon) = \lambda_0, \qquad \lim_{\varepsilon\searrow 0}\frac{\bar{\alpha}_\lambda}{\varepsilon^2} = \lambda_1.$$

We will find all homogenized equations and corresponding initial and boundary conditions for

$$\mu_0 = \lambda_0 = 0$$

in each of the following cases:

(1) $\mu_1 = \lambda_1 = \infty$;
(2) $0 \leqslant \lambda_1,\ \mu_1 < \infty$;
(3) $\mu_1 = \infty,\ 0 \leqslant \lambda_1 < \infty$;
(4) $\lambda_1 = \infty,\ 0 \leqslant \mu_1 < \infty$.

For example, to the case $\mu_1 = \lambda_1 = \infty$ corresponds the simple elliptic equation

$$\nabla\cdot\left(\frac{1}{\varrho(\boldsymbol{x})}\nabla p\right) = 0$$

for the pressure, while the conditions $0 \leqslant \lambda_1,\ \mu_1 < \infty$ lead to nonlocal homogenized equation

$$\nabla\cdot\left(\int_0^t \mathbb{B}(\mu_1, \lambda_1, t-\tau)\cdot\nabla p(\boldsymbol{x}, \tau)d\tau\right) = 0.$$

The justification of our results is based on a systematic use of the two-scale convergence method, which was proposed by G. Nguetseng [10] and has been recently used in a wide range of homogenization problems.