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On blow-up of solutions to differential inequalities with singularities on unbounded sets

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ABSTRACT

By the nonlinear capacity method, conditions of solvability are obtained for some classes of stationary and evolutional differential inequalities with coefficients that have singularities on unbounded sets.

1. Introduction

The problem of global existence versus blow-up of solutions to stationary and evolutional differential equations and inequalities with singular coefficients has been studied by many authors.

For the Laplacian and heat operator with a point singularity inside the domain, pioneering results in this direction were obtained by H. Brezis and X. Cabre [2] by means of comparison principles. Sufficient conditions of solvability for some elliptic problems with singularities on the boundary are due to A. Demyanov and A. Nazarov [3].

For higher order operators that do not satisfy the comparison principle, S. Pohozaev [5] suggested the nonlinear capacity method. Later it was developed in joint works with E. Mitidieri and other authors (see in particular the monograph [4] and references therein). This method allowed one to obtain a number of new sharp necessary conditions of global solvability of differential inequalities in various functional classes. The method is based on deriving asymptotically optimal a priori estimates of the solutions by means of algebraic analysis of the integral form of the inequality under consideration with a special choice of test functions.

In the present paper, a modification of the nonlinear capacity method is used in order to obtain necessary conditions of solvability for stationary differential inequalities and systems both of second and of higher order, including those with nonlinear principal terms and, most importantly, with coefficients having singularities on some unbounded sets inside the domain of definition. This distinguishes the problem setting suggested here from those of previous works in this field, where singularities appeared on certain bounded sets instead. Corresponding evolutional inequalities are also dealt with.

For the proof of nonexistence results by the nonlinear capacity method, test functions with different geometrical structures of the support are constructed, taking into account the specific natures of the problems under consideration.

The paper consists of six sections, where the problem of blow-up of solutions is studied for different classes of problems: in Section 2, for higher order semi-linear stationary inequalities; in Section 3, for stationary inequalities with nonlinear

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principal terms; in Section 4, for stationary inequalities with additional gradient terms; in Section 5, for systems of stationary inequalities; and in Section 6, for evolutional problems.

Remark on notation. From here on, letter c denotes different positive constants, which may depend on the parameters of the problems under consideration. Letters $c_1, c_2, \ldots$ denote absolute positive constants.

2. Semi-linear stationary inequalities

Let $S \subset \mathbb{R}^n$ be a closed, possibly unbounded set. Let $\varepsilon > 0$. Define

$$\rho(x) = \mathrm{dist}(x, S) = \inf\{|x - y| : y \in S\} \quad (x \in \mathbb{R}^n)$$

and

$$S^\varepsilon = \{x \in \mathbb{R}^n : \rho(x, S) < \varepsilon\}.$$

Assume that the following hypotheses hold:

(H1) There exist positive constants c_1, c_2, α such that for all $R > 0$ large enough and for all $\varepsilon > 0$ small enough one has

$$c_1\varepsilon^\alpha R^{n-\alpha} \le \mu((S^{3\varepsilon} \setminus S^{2\varepsilon}) \cap B_R(0)) \le \mu((S^{4\varepsilon} \setminus S^{\varepsilon}) \cap B_R(0)) \le c_2\varepsilon^\alpha R^{n-\alpha}. \tag{2.1}$$

(H2) There exists a family of functions $\xi_{\frac{1}{R}} \in C_0^{2k}(\mathbb{R}^n \setminus S; [0, 1])$ such that

$$\xi_{\frac{1}{R}}(x) = \begin{cases} 0 & (x \in S^{1/R} \cup (\mathbb{R}^n \setminus S^{4/R})), \\ 1 & (x \in S^{3/R} \setminus S^{2/R}). \end{cases} \tag{2.2}$$

Moreover, there exists a constant $c > 0$ such that

$$\left|D^\alpha \xi_{\frac{1}{R}}(x)\right| \le cR^{|\alpha|} \quad (x \in \mathbb{R}^n) \tag{2.3}$$

for all multi-indices α with $0 \le |\alpha| \le 2k$.

Further we also use functions $\psi_R \in C_0^{2k}(\mathbb{R}^n; [0, 1])$ such that

$$\psi_R(x) = \begin{cases} 1 & (|x| \le R), \\ 0 & (|x| \ge 2R) \end{cases} \tag{2.4}$$

and there exists a constant $c > 0$ such that

$$|D^\alpha \psi_R(x)| \le cR^{-|\alpha|} \quad (x \in \mathbb{R}^n) \tag{2.5}$$

for all multi-indices α with $0 \le |\alpha| \le 2k$.

Test functions $\varphi_R \in C_0^{2k}(\mathbb{R}^n \setminus S; [0, 1])$ are defined by

$$\varphi_R(x) = \xi_{\frac{1}{R}}^{\varkappa}(x)\psi_R^{\varkappa}(x) \tag{2.6}$$

with $\varkappa > 0$ large enough, as will be specified later.

Remark 2.1. The structure of $\xi_{\frac{1}{R}}$ and ψ_R is meant to get rid of singularities both on S and at infinity.

Existence of sets S that satisfy (H1) is shown in Examples 2.1 and 2.2. In Example 2.1, we also verify (H2).

Example 2.1. Consider the hyperplane $S = \Pi_n = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n: x_n = 0\}$. Then for all $R > 0$ large enough and $\varepsilon > 0$ small enough there holds

$$\mu(\Pi_n^\varepsilon \cap B_R(0)) = m_{n-1}R^{n-1}\varepsilon + o(R^{n-1}),$$

where m_{n-1} is the measure of the unit ball in $\mathbb{R}^{n-1}$. Thus Π_n satisfies condition (H1) with any $c_1 \in (0, m_{n-1})$, $c_2 > 3m_{n-1}$ and $\alpha = 1$. Similar estimates hold for any hyperplane in $\mathbb{R}^n$. For k-dimensional planes ($1 \le k \le n - 1$), condition (H1) holds with $\alpha = n - k$.

For $S = \Pi_n$, one can take $\xi_{\frac{1}{R}}(x) = \tilde{\xi}_{\frac{1}{R}}(x_n)$ and $\psi_R(x) = \tilde{\psi}_R(|x'|)$, where $|x'| = \sqrt{\sum_{i=1}^{n-1} x_i^2}$ and

$$\tilde{\xi}_{\frac{1}{R}}(x_n) = \begin{cases} 0 & (|x_n| \le 1/R \text{ or } |x_n| \ge 4/R), \\ 1 & (2/R \le |x_n| \le 3/R), \end{cases}$$

$$\tilde{\psi}_R(x') = \begin{cases} 1 & (x' \le R), \\ 0 & (x' \ge 2R). \end{cases}$$

In order to give an idea of C^{2k} functions $\tilde{\xi}_{\frac{1}{R}}(x_n)$ and $\tilde{\psi}_R(x')$, we show in Figs. 1 and 2 piecewise affine (C^0) functions $\xi^*_{\frac{1}{R}}(x_n)$ and $\psi^*_R(x')$ with similar structure.