J. Math. Anal. Appl. 408 (2013) 247–255

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Sharp decay rates for wave equations with a fractional damping via new method in the Fourier space

Ruy Coimbra Charão [a], Cleverson Roberto da Luz [a], Ryo Ikehata [b,*]

[a] Department of Mathematics, Federal University of Santa Catarina, Brazil

[b] Department of Mathematics, Graduate School of Education, Hiroshima University, Higashi-Hiroshima 739-8524, Japan

ARTICLE INFO

Article history:
Received 18 September 2012
Available online 17 June 2013
Submitted by David Russell

Keywords:
Wave equation
Fractional damping
Total energy decay
Multiplier method
Fourier analysis
Low frequency
High frequency
The Haraux–Komornik inequality

ABSTRACT

We study the Cauchy problem for damped wave equations with a fractional damping $(-\Delta)^\theta u_t$ in $\mathbf{R}^n$. We derive more sharp decay estimates of the total energy based on the energy method in the Fourier space combined with the Haraux–Komornik inequality. Especially, in the case when $0 \leq \theta \leq 1/2$ the rate of decay of the total energy becomes almost optimal. The method in this paper can be applied to other equations and in particular it seems to be quite effective in the case of frictional dissipation, i.e., when $\theta = 0$.

1. Introduction

We consider the initial value problem for the wave equation with fractional damping in $\mathbf{R}^n$:

$$u_{tt}(t,x) + Au(t,x) + A^\theta u_t(t,x) = 0, \quad (t,x) \in (0,\infty) \times \mathbf{R}^n \tag{1.1}$$

with initial data

$$u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x) \quad x \in \mathbf{R}^n, \tag{1.2}$$

where $A := -\Delta = -\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$.

The fractional power operator $A^\theta : \mathcal{D}(A^\theta) \subset L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$ $(\theta \geq 0)$ with its domain $\mathcal{D}(A^\theta) = H^{2\theta}(\mathbf{R}^n)$ is defined by

$$A^\theta v(x) := \mathcal{F}^{-1}\left(|\xi|^{2\theta}\mathcal{F}(v)(\xi)\right)(x), \quad v \in H^{2\theta}(\mathbf{R}^n), x \in \mathbf{R}^n,$$

where $\mathcal{F}$ denotes the usual Fourier transform in $L^2(\mathbf{R}^n)$ and $|\cdot|$ denotes the usual norm in $\mathbf{R}^n$. The operator A^θ is nonnegative and self-adjoint in $L^2(\mathbf{R}^n)$ and the Schwartz space $\mathcal{S}(\mathbf{R}^n)$ is dense in $H^{2\theta}(\mathbf{R}^n)$. Note that $A^1 = A$ and $A^0 = I$.

For each $(u_0, u_1) \in H^1(\mathbf{R}^n) \times L^2(\mathbf{R}^n)$ the problem (1.1)–(1.2) admits a unique mild solution $u \in \mathcal{C}([0,\infty); H^1(\mathbf{R}^n)) \cap \mathcal{C}^1([0,\infty); L^2(\mathbf{R}^n))$ provided that $\theta \geq 0$ (see Carvalho–Cholewa [2] and Lu–Reissig [9]).

We are concerned with the total energy decay estimates of solutions to problem (1.1)–(1.2). The Eq. (1.1) interpolates between the weak damping case ($\theta = 0$) and the strong damping case ($\theta = 1$). In the weak damping case we have historical

* Corresponding author.
E-mail addresses: charao@mtm.ufsc.br (R.C. Charão), cleverson@mtm.ufsc.br (C.R. da Luz), ikehatar@hiroshima-u.ac.jp (R. Ikehata).

0022-247X/$ – see front matter
http://dx.doi.org/10.1016/j.jmaa.2013.06.016

results due to Matsumura [10], while in the strong damping case we can cite the Ponce result [11] (for more precise results including exterior problems, see also Shibata [12] and Ikehata–Todorova–Yordanov [5]). Quite recently, Ikehata–Natsume [4] proved (by modifying the previous paper due to [5]) the following estimates to the total energy:

$$E_u(t) \le C(\|\nabla u_0\|^2 + \|u_1\|^2)e^{-\eta t} + C\|u_0\|^2_{L^1}(1+t)^{-\frac{n+2}{\alpha}} + \|u_1\|^2_{L^1}(1+t)^{-\frac{n}{\alpha}}, \tag{1.3}$$

where $\alpha := \max\{2-2\theta, 2\theta\}$, and $\eta > 0$ is a small constant. Note that Karch [6] has already derived similar decay estimates to (1.3) previously in the case when $\theta \in [0, 1/2]$. On the other hand, in the weak damping case $\theta = 0$ it follows from the Matsumura [10] result that

$$E_u(t) \le C(\|\nabla u_0\|^2 + \|u_1\|^2)e^{-\eta t} + C(\|u_0\|^2_{L^1} + \|u_1\|^2_{L^1})(1+t)^{-\frac{n+2}{2}}. \tag{1.4}$$

If we take $\theta = 0$ in (1.3), we have

$$E_u(t) \le C(\|\nabla u_0\|^2 + \|u_1\|^2)e^{-\eta t} + C\|u_0\|^2_{L^1}(1+t)^{-\frac{n+2}{2}} + \|u_1\|^2_{L^1}(1+t)^{-\frac{n}{2}}. \tag{1.5}$$

So, if we compare (1.4) with (1.5), we encounter a significant gap in the decay rates, i.e., the decay rate introduced in (1.3) cannot be connected continuously at $\theta = 0$. This shows that the rate of decay of (1.3) seems not to be optimal at least in the case when $\theta \in [0, 1/2]$. This is our motivation to re-study decay rates of the total energy. The approach which we use in this paper seems to be much different from the previous works due to Ponce [11], Lu–Reissig [9], Shibata [12], Karch [6], Ikehata–Natsume [4] and references therein. Our new method is relied on the energy method in the Fourier space (which has its origin in Umeda–Kawashima–Shizuta [13]) combined with the Haraux–Komornik inequality, the monotonicity of the localized and/or total energies in the Fourier space (see (3.8) and (3.9) below), and the property of $\mathbf{R}^n$ that power singularities less than n are integrable around the origin. This combination seems new. The method in this paper can be applied to the other equations and, in particular, it seems to be quite effective in the case of frictional dissipation, i.e., when $\theta = 0$. Applications to the other equations of our method will be announced in a forthcoming paper.

Notation. For $1 \le p \le \infty$, $L^p = L^p(\mathbf{R}^n)$ denotes the usual Lebesgue space with the norm $\|\cdot\|_{L^p}$. For simplicity of notations, in particular, we use $\|\cdot\|$ instead of $\|\cdot\|_{L^2}$. Let s be a nonnegative number, then $H^s = H^s(\mathbf{R}^n)$ denotes the usual Sobolev space of L^2 functions, equipped with the norm $\|\cdot\|_{H^s}$. Finally, in this paper, we write $u(t)$ instead of $u(t, x)$ in order to simplify the notation.

2. Results

The total energy $E_u(t)$ associated to the solution $u(t)$ of Eq. (1.1) is defined by

$$E_u(t) = \frac{1}{2}\left(\|u_t(t)\|^2 + \|\nabla u(t)\|^2\right).$$

It is very easy to see that $E_u(t)$ satisfies the following identity

$$E_u(t) + \int_0^t \|A^{\theta/2}u_t(s)\|^2\, ds = E_u(0), \tag{2.1}$$

for all $t \ge 0$.

Thus, the total energy is a non-increasing function of t. Our main result in this paper is given by the following theorem which shows explicit decay rates for the total energy depending on the power θ of the fractional damping and the dimension n.

Theorem 2.1. *Let $n \ge 1$ and $0 \le \theta \le 1$. If $[u_0, u_1] \in (H^1(\mathbf{R}^n) \cap L^1(\mathbf{R}^n)) \times (L^2(\mathbf{R}^n) \cap L^1(\mathbf{R}^n))$, then there exists a constant $C > 0$ and a constant $C_\beta > 0$ depending on β, such that the total energy associated to the solution $u(t, x)$ of (1.1)–(1.2) satisfies*

$$E_u(t) \le C_\beta\left\{\|u_0\|^2_{L^1} + \|u_1\|^2_{L^1}\right\} t^{-1/\beta} + C\left\{\|\nabla u_0\|^2 + \|u_1\|^2\right\} e^{-t/4}, \quad \forall t \ge T_0$$

where β is any positive fixed number satisfying

$$\beta > \frac{\alpha}{n - 2\theta + \alpha}$$

with $\alpha = \max\{2-2\theta, 2\theta\}$, and T_0 is a constant depending on the initial data.

We note that $\alpha \ge 1$ for $0 \le \theta \le 1$, $\alpha = 2-2\theta$ for $\theta \in [0, 1/2]$ and $\alpha = 2\theta$ for $\theta \in [1/2, 1]$.

Remark 2.1. The constants C_β and T_0 appear in the proof of the theorem (see (4.3) and (4.6)).

Remark 2.2. In the case of $\theta \in [0, 1/2)$, the decay rate just obtained in Theorem 2.1 becomes

$$E_u(t) = O(t^{-\frac{n+2-4\theta}{2(1-\theta)}+\delta}) \quad (t \to +\infty),$$