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On the roots of determinants of a class of holomorphic matrix-valued functions

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ABSTRACT

We consider matrix-valued functions that are holomorphic in the unit disk that are Cauchy transforms of finite matrix-valued measures. For non-zero roots z_j of the determinants of such functions, we provide estimates for $\sum(|z_j|^{-1}-1)$ and $\left[\sum(|z_j|^{-1}-1)^2\right]^{1/2}$.

1. Introduction

Let $\mathcal{H}(\mathbb{D})$ be the set of holomorphic functions in the unit disk $\mathbb{D}=\{z:|z|<1\}$. A classical result [3, Theorem 2.3] is that a function $f\in\mathcal{H}(\mathbb{D})$, $f\not\equiv 0$, belongs to the Nevanlinna class N of functions of bounded characteristic (i.e., $1/2\pi\int_0^{2\pi}|\log|f(re^{i\theta})||d\theta<\infty$) if and only if its roots z_j, $|z_j|<1$, satisfy the Blaschke condition

$$\sum_j(1-|z_j|)<\infty. \tag{1}$$

Class N is quite wide and contains, in particular, classical Hardy spaces $H^p(\mathbb{D})$, $p>0$. Moreover, any function $f\in\mathcal{H}(\mathbb{D})$ that can be represented as the Cauchy transform of a finite complex-valued Borel measure on the unit circle $\mathbb{T}$ belongs to N. Since class N is an algebra, it is clear that if $F(z)=(F_{jk}(z))_{j,k=1}^n$ is an $n\times n$ matrix-valued function with entries $F_{jk}\in N$, then $\det(F)$ is also a function of class N.

Even though condition (1) is well known, I have not found any actual estimate of the sum on the left-hand side of (1) in terms of the function. Estimates of the Blaschke sum $\sum(1-|z_j|)$ have been studied in terms of the so-called concentration [1,5,9].

It is clear that for nonzero roots z_j of function $f\in N$, conditions (1) and

$$\sum_j(|z_j|^{-1}-1)<\infty \tag{2}$$

are equivalent. Hansmann and Katriel estimated (2) for functions $f\in\mathcal{H}(\mathbb{D})$ $(f(0)\neq 0)$ that are representable as Cauchy transforms of finite Borel measures on $\mathbb{T}$, and in particular for $f\in H^p(\mathbb{D})$, $p\geq 1$ [8]. The approach they utilized was pure operator-theoretical.

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Here I adopt ideas similar to those used by Hansmann and Katriel [8] and provide estimates of the sum on the left-hand side of (2) and for

$$\left[\sum_j (|z_j|^{-1}-1)^2\right]^{1/2}, \tag{3}$$

where $\{z_j\}$ are roots of $\det G(z)$, $\det G(0) \neq 0$, and $G(z)$ is an $n \times n$ matrix-valued holomorphic function that is the Cauchy transform of an $n \times n$ matrix-valued measure. In particular, I obtain estimates for matrix-valued functions from $H^2_{n\times n}(\mathbb{D})$, the set of all $n \times n$ matrix-valued functions with entries from $H^2(\mathbb{D})$. To the best of my knowledge, these results are new. In the scalar-valued case ($n = 1$), our estimate for the left-hand side of (2) coincides with that in [8].

The remainder of the paper is organized as follows. In Section 2 we formulate some properties of the Cauchy transform of finite Borel complex-valued measures and discuss properties of the Cauchy transform of matrix-valued measures. In Section 3, using an $n \times n$ matrix-valued finite measure, we construct a Hilbert space and a finite-dimensional operator, and estimate the trace and Hilbert–Schmidt norms of that operator. In Section 4 we introduce a unitary operator and consider its perturbation by the finite-dimensional operator constructed in Section 3. We find that the perturbation determinant is closely related to the determinant of the Cauchy transform of the matrix-valued measure. We also prove some estimates for eigenvalues of the perturbed operator. These estimates, along with those obtained in Section 3, allow us to prove the main result (Theorem 1), Corollary 1, and a version of Theorem 1 for matrix-valued functions from $H^2_{n\times n}(\mathbb{D})$ (Theorem 2).

2. The Cauchy transform

In this section we provide information about the Cauchy transform. A detailed treatment of the Cauchy transform is available in the literature [2].

Let $\mathcal{H}(\mathbb{D})$ be the class of all functions that are holomorphic in the unit disk $\mathbb{D}$ and let $\mathcal{M}$ be the class of all finite Borel measures on the unit circle $\mathbb{T}$. For $\mu \in \mathcal{M}$, let $\mathcal{C}\mu$ be defined as follows:

$$(\mathcal{C}\mu)(z) = \int_{\mathbb{T}} \frac{d\mu(\zeta)}{1 - z\bar{\zeta}}, \quad z \in \mathbb{D}. \tag{4}$$

The function $\mathcal{C}\mu$ is in $\mathcal{H}(\mathbb{D})$ and is called the Cauchy transform of the measure μ. The vector space of all Cauchy transforms of measures from $\mathcal{M}$ is denoted by $\mathfrak{C}\mathcal{M}$, that is,

$$\mathfrak{C}\mathcal{M} = \{h \in \mathcal{H}(\mathbb{D}) : h = \mathcal{C}\mu \text{ for some } \mu \in \mathcal{M}\}.$$

The space $\mathfrak{C}\mathcal{M}$ is a Banach space with norm

$$\|h\|_{\mathfrak{C}\mathcal{M}} = \inf\{\|\mu\| : \mu \in \mathcal{M},\ \mathcal{C}\mu = h\}, \tag{5}$$

where $\|\mu\|$ is the total variation of the measure μ. Moreover, for each $h \in \mathfrak{C}\mathcal{M}$ there exists a unique $\mathring{\mu} \in \mathcal{M}$ such that

$$\mathcal{C}\mathring{\mu} = h \quad \text{and} \quad \|h\|_{\mathfrak{C}\mathcal{M}} = \|\mathring{\mu}\|. \tag{6}$$

Later we call such a $\mathring{\mu}$ the minimal measure and we call the representation of a function $h \in \mathfrak{C}\mathcal{M}$ as the Cauchy transform of the minimal measure the minimal representation.

It is known that

$$\bigcup_{p\geq 1} H^p(\mathbb{D}) \subset \mathfrak{C} \subset \bigcap_{0<p<1} H^p(\mathbb{D}) \tag{7}$$

(both inclusions are strict) and

$$\|h\|_{\mathfrak{C}\mathcal{M}} \leq \|h\|_{H^p}, \quad h \in H^p(\mathbb{D}),\ p \geq 1. \tag{8}$$

From (7) it follows that any $h \in \mathfrak{C}\mathcal{M}$ also belongs to the Nevanlinna class N of functions of bounded characteristic.

For $h \in \mathcal{H}(\mathbb{D})$ the backward shift operator B is defined as

$$(Bh)(z) = \frac{h(z) - h(0)}{z}, \quad z \in \mathbb{D}. \tag{9}$$

The following two statements are important.

Lemma 1. *Let $h \in \mathcal{H}(\mathbb{D})$. Then $h \in \mathfrak{C}\mathcal{M}$ if and only if $Bh \in \mathfrak{C}\mathcal{M}$.*

Lemma 2. *The backward shift operator $B : (\mathfrak{C}, \|\cdot\|_{\mathfrak{C}\mathcal{M}}) \to (\mathfrak{C}, \|\cdot\|_{\mathfrak{C}\mathcal{M}})$ is bounded and has norm 1, that is,*

$$\|B\|_{\mathfrak{C}\mathcal{M}} = \sup_{h\in\mathfrak{C}\mathcal{M}, h\neq 0} \frac{\|Bh\|_{\mathfrak{C}\mathcal{M}}}{\|h\|_{\mathfrak{C}\mathcal{M}}} = 1.$$

Lemma 1 [8, Lemma 1] and Lemma 2 [2, Proposition 11.3.1] have been proved elsewhere.