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Existence of three solutions for a first-order problem with nonlinear nonlocal boundary conditions

Douglas R. Anderson

Department of Mathematics, Concordia College, Moorhead, MN 56562, USA

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ABSTRACT

Conditions for the existence of at least three positive solutions to the nonlinear first-order problem with a nonlinear nonlocal boundary condition given by

$$y'(t) - r(t)y(t) = \sum_{i=1}^{m} f_i\Big(t, y(t)\Big), \quad t \in [0, 1],$$

$$\lambda y(0) = y(1) + \sum_{j=1}^{n} \Lambda_j(\tau_j, y(\tau_j)), \quad \tau_j \in [0, 1],$$

are discussed, for sufficiently large $\lambda > 1$ and $r \geq 0$. The Leggett–Williams fixed point theorem is utilized.

1. Introduction

We are interested in the first-order boundary value problem with a nonlinear nonlocal boundary condition given by

$$y'(t) - r(t)y(t) = \sum_{i=1}^{m} f_i\left(t, y(t)\right), \quad t \in [0, 1], \tag{1.1}$$

$$\lambda y(0) = y(1) + \sum_{j=1}^{n} \Lambda_j(\tau_j, y(\tau_j)), \quad \tau_j \in [0, 1], \tag{1.2}$$

where $r : [0, 1] \to [0, \infty)$ is continuous; the nonlocal points satisfy $0 \leq \tau_1 < \tau_2 < \cdots < \tau_n \leq 1$; the nonlinear functions $\Lambda_j : [0, 1] \times [0, \infty) \to [0, \infty)$ satisfy

$$0 \leq y\psi_j(t, y) \leq \Lambda_j(t, y) \leq y\Psi_j(t, y), \quad t \in [0, 1], y \in [0, \infty), \tag{1.3}$$

for some positive continuous (possibly nonlinear) functions $\psi_j, \Psi_j : [0, 1] \times [0, \infty) \to [0, \infty)$; the scalar λ satisfies

$$\lambda > \left(1 + \sum_{j=1}^{n} \beta_j\right) \exp\left(\int_0^1 r(\eta) d\eta\right) > 1, \quad \beta_j := \max_{[0,1]\times[0,C]} \Psi_j(t, y) \tag{1.4}$$

for some real constant $C > 0$; and the nonlinear functions $f_i : [0, 1] \times [0, \infty) \to [0, \infty)$ are all continuous. We also set

$$\alpha_j := \min_{[0,1]\times[B,\lambda B]} \psi_j(t, y) \tag{1.5}$$

for some constant real $B > 0$ for later reference. Note that by continuity and compactness α_j and β_j exist and satisfy $\beta_j > \alpha_j > 0$.

E-mail address: andersod@cord.edu.

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Some of the motivation for this paper and the study of problem (1.1), (1.2) is as follows. First-order equations with various boundary conditions, including multi-point and nonlocal conditions, are of recent interest. For example, we cite the following papers. Zhao and Sun [14] were concerned with the first-order PBVP (if $\mathbb{T}=\mathbb{R}$)

$$y'(t)+r(t)y(t)=\lambda f\left(t,y(t)\right), \quad t\in[0,1], \tag{1.6}$$
$$y(0)=y(1). \tag{1.7}$$

Tian and Ge [12] investigated the first-order three-point problem (if $\mathbb{T}=\mathbb{R}$)

$$y'(t)+r(t)y(t)=\lambda f\left(t,y(t)\right), \quad t\in[0,1], \tag{1.8}$$
$$y(0)-\alpha y(\eta)=\gamma y(1), \tag{1.9}$$

while Gao and Luo [2] were interested in the problem (if $\mathbb{T}=\mathbb{R}$)

$$y'(t)+r(t)y(t)=\lambda f\left(t,y(t)\right), \quad t\in[0,1], \tag{1.10}$$
$$y(0)=\sum_{j=1}^{n}\gamma_j y(t_j); \tag{1.11}$$

similarly Anderson [1] studied the first-order problem (if $\mathbb{T}=\mathbb{R}$)

$$y'(t)+r(t)y(t)=\lambda f\left(t,y(t)\right), \quad t\in[0,1], \tag{1.12}$$
$$y(0)=y(1)+\sum_{j=1}^{n}\gamma_j y(t_j). \tag{1.13}$$

In a related paper, Nieto and R. Rodríguez-López [8] considered

$$y'(t)+r(t)y(t)=\lambda f\left(t\right), \quad t\in[0,1], \tag{1.14}$$
$$\lambda y(t_0)=\sum_{j=1}^{n}\gamma_j y(t_j). \tag{1.15}$$

Gilbert [3] looked at (if $\mathbb{T}=\mathbb{R}$)

$$y'(t)=\lambda f\left(t,y(t)\right), \quad a.e.\, t\in[0,1], \tag{1.16}$$
$$y(0)=y(1), \quad \text{or} \quad y(0)=y_0 \tag{1.17}$$

using measure theory and Δ-Carathéodory functions. Goodrich [4] analyzed the p-Laplacian problem (if $\mathbb{T}=\mathbb{R}$)

$$\phi_p\left(y'(t)\right)=h(t)f\left(y(t)\right), \quad t\in[0,1], \tag{1.18}$$
$$y(0)=\Psi(y) \quad \text{or} \quad y(0)=B_0\left(y'(1)\right) \quad \text{or} \quad y(0)=\left(y'(1)\right)^m, \tag{1.19}$$

while Graef and Kong [6] explored the related p-Laplacian problem (if $\mathbb{T}=\mathbb{R}$)

$$\phi_p\left(y'(t)\right)=f\left(t,y(t)\right), \quad t\in[0,1], \tag{1.20}$$
$$y(0)=B\left(y'(1)\right). \tag{1.21}$$

Otero-Espinar and Vivero [9] gave a general view of different kinds of weak first-order discontinuous boundary value problems on an arbitrary time scale with nonlinear functional boundary value conditions. Shu and Deng [11] proved the existence of three positive solutions to (if $\mathbb{T}=\mathbb{R}$)

$$y'(t)=\lambda f\left(y(t)\right), \quad t\in[0,1], \tag{1.22}$$
$$y(0)=\gamma y(1). \tag{1.23}$$

Zhao [13] applied a monotone iteration method to the problem (if $\mathbb{T}=\mathbb{R}$)

$$y'(t)+r(t)y(t)=\lambda f\left(t,y(t)\right), \quad t\in[0,1], \tag{1.24}$$
$$y(0)=g(y(1)), \tag{1.25}$$

where g denotes a nonlinear boundary condition. Precup and Trif [10] dealt with the existence, localization and multiplicity of positive solutions to nonlocal problems for first order differential systems of the form

$$y'(t)=f\left(t,y(t)\right), \quad t\in[0,1], \tag{1.26}$$
$$y(0)=\alpha[y], \tag{1.27}$$