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Periods of Jacobi forms and the Hecke operator

YoungJu Choie *, Seokho Jin

Department of Mathematics and PMI, Pohang University of Science and Technology, Pohang, 790-784, Republic of Korea

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ABSTRACT

A Hecke action on the space of periods of cusp forms, that is compatible with that on the space of cusp forms, was first computed using continued fraction [20] and an explicit algebraic formula of Hecke operators acting on the space of period functions of modular forms was derived by studying the rational period functions [9]. As an application an elementary proof of the Eichler–Selberg trace formula was derived [27]. A similar modification has been applied to the space of period functions of Maass cusp forms with spectral parameter s [22,23,21]. In this paper we study the space of period functions of Jacobi forms by means of the Jacobi integral and give an explicit description of the action of Hecke operators on this space. A Jacobi–Eisenstein series $E_{2,1}(\tau, z)$ of weight 2 and index 1 is discussed as an example. Periods of Jacobi integrals already appeared in a disguised form in the work of Zwegers in his study of the Mordell integral coming from Lerch sums [28], and mock Jacobi forms are typical examples of the Jacobi integral [10].

1. Introduction

Period functions of modular forms have played an important role to understand the arithmetics on cusp forms [18]. Manin [20] studied a Hecke action on the space of periods of cusp forms, which is compatible with that on the space of cusp forms, in terms of continued fractions. Later an explicit algebraic description of a Hecke operator on the space of period functions of modular forms was given by studying the rational period functions [9]. As an application, a new elementary proof of the Eichler–Selberg trace formula was derived [26,27]. Moreover, various modifications of the theory of period functions have been also developed [13]. A notion of rational period functions has been introduced and completely classified in the case of cofinite subgroups of $SL_2(\mathbb{Z})$ [15,17,16,2]. Some period functions were attached bijectively to Maass cusp forms according to the spectral parameter s and its cohomological counterpart was described (see [19,3]). Similar modifications to Hecke operators on the period space have also been applied to that of Maass cusp forms with spectral parameter s [22,23,21].

Historically, Eichler [11] and Shimura [25] discovered an isomorphism between a space of cusp forms and the Eichler cohomology group, attaching period polynomials to cusp forms. Knopp [15] introduced the notion of Eichler integral for arbitrary real weight with multiplier system and showed that there is an isomorphism between the space of modular forms and Eichler cohomology group. Along this line, recently a notion of Jacobi integral has been introduced [8]. It was shown that there is also an isomorphism between the space of Jacobi cusp forms and the corresponding Eichler cohomology group [4]. The mock Jacobi form [10] is one of the typical example of the Jacobi integral.

The main purpose of this paper is to give an explicit algebraic description of Hecke operators on the period functions attached to the Jacobi integrals. This result is analogous to the result in [9] for the case of Jacobi forms.

This paper is organized as follows. In Section 2 we state the main results and Section 3 discusses a Jacobi–Eisenstein series $E_{2,1}(\tau, z)$ as an example of the Jacobi integral with period functions. Basic definitions and notations are given in Section 4. Section 5 gives proofs of the main theorems by introducing various properties which are modification of the results concerning rational period functions [9].

* Corresponding author.
E-mail addresses: yjc@postech.ac.kr, yjchoie@gmail.com (Y. Choie), archimed@postech.ac.kr (S. Jin).

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2. Statement of main results

We recall known facts concerning Jacobi forms [12] and Jacobi integrals. We use the standard notation that will be recalled in Section 4.

Let f be an element of $J^{\int}_{k,m}(\Gamma(1))$, that is, f is a real analytic function $f:\mathbb{H}\times\mathbb{C}\to\mathbb{C}$ satisfying a certain growth condition, with the following functional equation, for any element γ in the Jacobi group $\Gamma(1)^J$ (see Section 4 for the definition),

$$(f|_{k,m}\gamma)(\tau,z)=f(\tau,z)+P_\gamma(\tau,z) \tag{2.1}$$

with $P_\gamma\in\mathcal{P}_m$, where $\mathcal{P}_m$ is a space of holomorphic functions

$$\mathcal{P}_m=\left\{g:\mathbb{H}\times\mathbb{C}\to\mathbb{C}|\,|g(\tau,z)|<K(|\tau|^\rho+v^{-\sigma})e^{2\pi m\frac{y^2}{v}},\ \text{for some } K,\rho,\sigma>0\right\} \tag{2.2}$$

($v=Im(\tau)$ and $y=Im(z)$).

P_γ in (2.1) is called a *period function* of f. If $P_\gamma(\tau,z)=0,\forall\gamma\in\Gamma(1)^J$, f is a usual Jacobi form if f is holomorphic [12], more generally, a real analytic Jacobi form [1,8,24]. Using the generators and relations among the elements of Jacobi group $\Gamma(1)^J$, it turns out that each element of the following set

$$\mathcal{P}er_{k,m}:=\left\{P:\mathbb{H}\times\mathbb{C}\to\mathbb{C}\left|\sum_{j=0}^{3}P\right|_{k,m}T^j=\sum_{j=0}^{5}P|_{k,m}U^j=0\right\} \tag{2.3}$$

$\left(T=\left[\begin{pmatrix}0&-1\\1&0\end{pmatrix},(1,0)\right],S=\left[\begin{pmatrix}1&1\\0&1\end{pmatrix},(0,0)\right],U=ST\right)$ generates a system of period functions $\{P_\gamma|\gamma\in\Gamma(1)^J\}$ of $f\in J^{\int}_{k,m}(\Gamma(1))$ (see Section 4 for details).

We also consider the following subspace:

$$\mathcal{E}J^{\int}_{k,m}(\Gamma(1)):=\left\{f\in J^{\int}_{k,m}(\Gamma(1))|f|_{k,m}[I,(1,0)]=f\right\}. \tag{2.4}$$

Putting further conditions on the elliptic elements in $\Gamma(1)^J$, it turns out that the following set

$$\mathcal{E}\mathcal{P}er_{k,m}:=\left\{P:\mathbb{H}\times\mathbb{C}\to\mathbb{C}\left|\sum_{j=0}^{3}P\right|_{k,m}T^j=\sum_{j=0}^{5}P|_{k,m}U^j=P-P|_{k,m}[I,(1,0)]=0\right\} \tag{2.5}$$

is a generating set of a system of period functions $\{P_\gamma\,|\,\gamma\in\Gamma(1)^J\}$ of $f\in\mathcal{E}J^{\int}_{k,m}(\Gamma(1))$ (see Section 4).

Now let us recall the definition of two Hecke operators on the space of Jacobi forms studied by Eichler and Zagier [12, Section 4]: for each positive integer n consider

(1) $\left(f|_{k,m}V_n\right)(\tau,z):=n^{k-1}\sum_{\substack{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\Gamma_1\backslash M_2(\mathbb{Z})\\ ad-bc=n}}(c\tau+d)^{-k}e^{-2\pi imn\frac{cz^2}{c\tau+d}}f\left(\frac{a\tau+b}{c\tau+d},\frac{lz}{c\tau+d}\right)$,

(2) $(f|_{k,m}T_n)(\tau,z):=n^{k-4}\sum_{\substack{\gamma\in\Gamma\backslash M_2(\mathbb{Z}),X,Y\in\mathbb{Z}/n\mathbb{Z}\\ \det\gamma=n^2,\gcd(\gamma)=\square}}f|_{k,m}\left[\begin{pmatrix}a&b\\c&d\end{pmatrix},(X,Y)\right]$,

where $f|_{k,m}\left[\begin{pmatrix}a&b\\c&d\end{pmatrix},(X,Y)\right]:=f|_{k,m}\left[\begin{pmatrix}a/n&b/n\\c/n&d/n\end{pmatrix},(X,Y)\right]$ if $ad-bc=n^2$.

It was shown [12, Theorem 4.1] that the operators V_n, T_n are well defined on $J_{k,m}(\Gamma(1))$, the space of Jacobi forms of weight k and index m, and map $J_{k,m}(\Gamma(1))$ to $J_{k,mn}(\Gamma(1))$ and $J_{k,m}(\Gamma(1))$, respectively. Since a general Jacobi integral $f\in J^{\int}_{k,m}(\Gamma(1))$ is not $\Gamma(1)^J$-invariant, these operators V_n and T_n do not act on $J^{\int}_{k,m}(\Gamma(1))$.

So one needs to choose a special set of representatives to apply the Hecke operator to a Jacobi integral (see Section 5 for details): for each positive integer n, define two Hecke operators $\mathcal{V}_n^\infty$ and $\mathcal{T}_n^\infty$ by

$$\left(f|_{k,m}\mathcal{V}_n^\infty\right)(\tau,z):=n^{k-1}\sum_{\substack{ad=n,a>o\\ b(\mathrm{mod}\ d)}}d^{-k}f\left(\frac{a\tau+b}{d},az\right) \tag{2.6}$$

$$(f|_{k,m}\mathcal{T}_n^\infty)(\tau,z):=n^{k-4}\sum_{\substack{ad=n^2,\gcd(a,b,d)=\square\\ a,d>0,b(\mathrm{mod}\ d),X,Y\in\mathbb{Z}/n\mathbb{Z}}}\left(f|_{k,m}\left[\begin{pmatrix}a&b\\0&d\end{pmatrix},(X,Y)\right]\right)(\tau,z). \tag{2.7}$$