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Von Neumann entropy and majorization

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ABSTRACT

We consider the properties of the Shannon entropy for two probability distributions which stand in the relationship of majorization. Then we give a generalization of a theorem due to Uhlmann, extending it to infinite dimensional Hilbert spaces. Finally we show that for any quantum channel Φ, one has $S(\Phi(\rho)) = S(\rho)$ for all quantum states ρ if and only if there exists an isometric operator V such that $\Phi(\rho) = V\rho V^*$.

1. Introduction

In this paper, we study aspects of the relation of majorization for classical probability distributions and for quantum mechanical density operators and connections with the action of quantum channels, utilizing relevant properties of the Shannon entropy and von Neumann entropy. We obtain extensions of classic results from finite to infinite dimensional Hilbert spaces.

Let $\mathcal{B}(\mathcal{H})$ be the von Neumann algebra of all bounded linear operators on a separable Hilbert space $\mathcal{H}$ over $\mathbb{C}$ and $S(\mathcal{H})$ be the set of all *density operators* on $\mathcal{H}$. That is, $\rho \in S(\mathcal{H})$ if and only if $\rho \geq 0$ and $\mathrm{tr}(\rho) = 1$. The elements of $S(\mathcal{H})$ are taken to represent quantum states in quantum physics, while the self-adjoint elements of $\mathcal{B}(\mathcal{H})$ represent (bounded) observables. The set $S(\mathcal{H})$ spans the Banach space $\mathcal{T}(\mathcal{H})$ of all trace class operators on $\mathcal{H}$.

We denote by $\mathcal{B}(\mathcal{H}, \mathcal{K})$ the space of all bounded linear operators from $\mathcal{H}$ into another Hilbert space $\mathcal{K}$. An operator $V \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is called an isometry if $V^*V = I_{\mathcal{H}}$. In this case, $VV^* \in \mathcal{B}(\mathcal{K})$ is an orthogonal projection. For $x, y \in \mathcal{H}$, $x \otimes y$ denotes the (linear, rank-1) operator $z \mapsto \langle z, y\rangle x (z \in \mathcal{H})$. For each $\rho \in S(\mathcal{H})$ we denote by $\lambda(\rho) \equiv (\lambda_1(\rho), \lambda_2(\rho), \ldots \lambda_n(\rho) \cdots)$ the sequence of eigenvalues of ρ, arranged in non-increasing order. Thus $\lambda(\rho) \in c_0^*$, where c_0^* is the positive cone of sequences decreasing monotonically to 0, as defined in [7].

Let $l^\infty(\mathbb{R})$ and $l_1^1(\mathbb{R}^+)$ denote the sets of all bounded real sequences and of all summable non-negative real sequences which have sum 1, respectively. For a vector $r \in l^\infty(\mathbb{R})$, we introduce $r^\downarrow = (r_1^\downarrow, \ldots, r_n^\downarrow \cdots)$ as the vector whose elements are the elements of r re-ordered into non-increasing order. Adopting the definition of majorization given in [2,7], for $r, s \in c_0^*$, we say that r is majorized by s, written as $r \prec s$, if

$$\sum_{i=1}^k r_i^\downarrow \leq \sum_{i=1}^k s_i^\downarrow, \quad \text{for } k = 1, 2, \ldots \quad \text{and} \quad \sum_{i=1}^\infty r_i^\downarrow = \sum_{i=1}^\infty s_i^\downarrow.$$

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Recently, the infinite dimensional Schur–Horn theorem and infinite majorization have received much attention. In [12], A. Neumann has given properties of infinite majorization in $l^\infty(\mathbb{R})$; V. Kaftal and G. Weiss [7] have obtained interesting results for infinite majorization in c_0^*. Arveson and Kadison [3] presented some other characterizations by different methods.

Some results relevant for our purposes are the following: if $r, s \in l_1^1(\mathbb{R}^+)$, then

$$r \prec s \Longleftrightarrow r = Qs, \quad \text{with } Q_{ij} = |U_{ij}|^2 \text{ for some unitary } U \text{ [4, Theorem 1]},$$

and

$$r \prec s \Longleftrightarrow r = Qs, \quad \text{for some orthostochastic matrix } Q \text{ [7, Corollary 6.1]}.$$

Motivated by the above studies, we first consider the properties of the Shannon entropy for two elements in $l_1^1(\mathbb{R}^+)$ that satisfy majorization. Then we extend and study those properties for two density operators $\rho, \sigma \in S(\mathcal{H})$. Following the definition given for finite dimensional spaces (see [1]), we denote $\rho \prec \sigma$ for two operators $\rho, \sigma \in S(\mathcal{H})$ if $\lambda(\rho) \prec \lambda(\sigma)$.

Let $\mathcal{M}_n(\mathcal{B}(\mathcal{H}))$ be the von Neumann algebra of $n \times n$ matrices whose entries are in $\mathcal{B}(\mathcal{H})$, and let $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a linear map. Then Φ induces a map $\Phi_n : \mathcal{M}_n(\mathcal{B}(\mathcal{H})) \to \mathcal{M}_n(\mathcal{B}(\mathcal{H}))$ by the formula

$$\Phi_n((a_{i,j})) = (\Phi(a_{i,j})) \quad \text{for } (a_{i,j}) \in \mathcal{M}_n(\mathcal{B}(\mathcal{H})).$$

If every Φ_n is a positive map, then Φ is called completely positive. Φ is said to be normal if Φ is continuous with respect to the ultraweak (σ-weak) topology. Normal completely positive contractive maps on $\mathcal{B}(\mathcal{H})$ were characterized in a theorem of Kraus [8, Theorem 3.3], which says that Φ is a normal contractive completely positive map if and only if there exists a sequence $\{A_i\}_{i=1}^\infty$ in $\mathcal{B}(\mathcal{H})$ such that for all $X \in \mathcal{B}(\mathcal{H})$,

$$\Phi(X) = \sum_{i=1}^\infty A_i X A_i^* \quad \text{with} \quad \sum_{i=1}^\infty A_i A_i^* \le I$$

where the limits are defined in the strong operator topology. The sequence $\{A_i\}_{i=1}^\infty$, which is not unique, is also called a family of Kraus operators for Φ. In this case, Φ has a dual map $\Phi^\dagger$ defined by

$$\Phi^\dagger(X) = \sum_{i=1}^\infty A_i^* X A_i \quad \text{for } X \in \mathcal{T}(\mathcal{H}),$$

where the sum converges in the trace norm topology. It is easy to see that one has

$$|\mathrm{tr}[\Phi^\dagger(X)Y]| = |\mathrm{tr}[X\Phi(Y)]| \le \|\Phi\| \, \|Y\| \, \mathrm{tr}(|X|), \quad X \in \mathcal{T}(\mathcal{H}), Y \in \mathcal{B}(\mathcal{H}),$$

so $\Phi^\dagger(X) \in \mathcal{T}(\mathcal{H})$ and $\Phi^\dagger$ is well defined on $\mathcal{T}(\mathcal{H})$. In general, $\Phi^\dagger$ cannot be extended from $\mathcal{T}(\mathcal{H})$ into $\mathcal{B}(\mathcal{H})$. However, if $\sum_{i=1}^\infty A_i^* A_i \le I$, then $\Phi^\dagger$ is well defined as a normal map on $\mathcal{B}(\mathcal{H})$. A normal completely positive map Φ which is trace preserving ($\Phi^\dagger(I) = I$, corresponding to $\mathrm{tr}\,(\Phi(X)) = \mathrm{tr}(X)$ for $X \in \mathcal{T}(\mathcal{H})$) is called a *quantum channel*. If a normal completely positive map satisfies $\Phi(I) \le I$, then Φ is called a quantum operation [8,9]. A quantum operation is unital if $\Phi(I) = I$, which is equivalent to $\sum_j A_j A_j^* = I$. A quantum operation is bi-stochastic if it is both trace-preserving and unital. In particular, Φ is said to be a mixed unitary operation if $\Phi(X) = \sum_{i=1}^n t_i U_i X U_i^*$, where $n < \infty$, the U_i are all unitary operators and $t_i > 0$, $\sum_{i=1}^n t_i = 1$.

The von Neumann entropy of a quantum state ρ is defined by the formula

$$S(\rho) \equiv -\mathrm{tr}(\rho \log(\rho)).$$

Here we follow the common practice of taking the logarithm to base two. In classical information theory, the Shannon entropy is defined by $H(p) = -\sum_i p_i \log(p_i)$, where $p = (p_1, p_2 \cdots p_n \cdots)$ is a probability distribution. If λ_i are the eigenvalues of ρ, then the von Neumann entropy can be re-expressed as

$$S(\rho) = -\sum_{i=1}^\infty \lambda_i \log(\lambda_i) = H(\lambda(\rho)),$$

where we use $0 \log 0 = 0$.

There are extensive recent studies of quantum operations which preserve the von Neumann entropy and the relative entropy of quantum states [6,10,15,17].

Hardy, Littlewood and Pólya [5] showed for $\xi, \eta \in \mathbb{R}^n$,

$$\xi \prec \eta \Longleftrightarrow \xi = Q\eta \quad \text{for some doubly stochastic matrix } Q.$$

In the quantum context, Uhlmann proved the following for any pair of density operators $\rho, \sigma \in S(\mathcal{H})$ acting in a finite dimensional Hilbert space $\mathcal{H}$:

$$\rho \prec \sigma \Longleftrightarrow \rho = \Phi(\sigma), \quad \text{for some mixed unitary quantum operation } \Phi.$$

Uhlmann's theorem can be used to study the role of majorization in quantum mechanics.

Here we first consider the properties of the Shannon entropies of two probability distributions which obey majorization. Then we give a generalization of Uhlmann's theorem for infinite dimensional Hilbert spaces. Finally, we give a characterization of quantum channels that leave the von Neumann entropy invariant.