J. Math. Anal. Appl. 410 (2014) 11–26

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Series representations in the spirit of Ramanujan

Emre Alkan

Department of Mathematics, Koç University, Rumelifeneri Yolu, 34450, Sarıyer, Istanbul, Turkey

ARTICLE INFO

Article history:
Received 10 December 2012
Available online 16 August 2013
Submitted by B.C. Berndt

Keywords:
Integral transform
Trigonometric functions
Hyperbolic functions
Central binomial coefficient
Catalan's constant
Geometric rate of convergence
L-functions
Riemann zeta function
Hurwitz zeta function
Bernoulli polynomials
Bernoulli numbers

ABSTRACT

Using an integral transform with a mild singularity, we obtain series representations valid for specific regions in the complex plane involving trigonometric functions and the central binomial coefficient which are analogues of the types of series representations first studied by Ramanujan over certain intervals on the real line. We then study an exponential type series rapidly converging to the special values of *L*-functions and the Riemann zeta function. In this way, a new series converging to Catalan's constant with geometric rate of convergence less than a quarter is deduced. Further evaluations of some series involving hyperbolic functions are also given.

1. Introduction

Our purpose in this paper is twofold. First we introduce an integral transform having a mild singularity and in particular generalizing the classical representation (see [23])

$$Li_n(z) := \sum_{k=1}^{\infty} \frac{z^k}{k^n} = \frac{(-1)^{n-1}}{(n-1)!} \int_0^1 \frac{z \log^{n-1} x}{1-zx}\, dx,$$

for the polylogarithm function of order $n \geqslant 2$, where z is a complex number with $|z| \leqslant 1$. With the help of our transform, we then show how to find analogues of certain series representations involving trigonometric functions and the central binomial coefficient. Such type of series were first studied by Ramanujan [24,25]. We refer the reader to the monograph of Berndt [15] which gives further enlightening discussions on the nature of these representations. Ramanujan proved for example that

$$\sum_{n=0}^{\infty} \frac{(2\sin 2x)^{2n+1}}{\binom{2n}{n}(2n+1)^2} = 2\sum_{n=0}^{\infty} \frac{(\tan x)^{2n+1}}{(2n+1)^2},$$

for any real number x with $|x| \leqslant \frac{\pi}{4}$. In this way, Ramanujan obtained striking rapidly convergent series for the Catalan constant and $\zeta(3)$, where $\zeta(s)$ is the Riemann zeta function. Concerning Catalan's constant, he showed that (see [24,25,15,18])

E-mail address: ealkan@ku.edu.tr.

0022-247X/$ – see front matter
http://dx.doi.org/10.1016/j.jmaa.2013.08.021

$$G:=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}=\frac{\pi}{8}\log(2+\sqrt{3})+\frac{3}{8}\sum_{n=0}^{\infty}\frac{1}{\binom{2n}{n}(2n+1)^2}.$$

In recent work, Batır [11] obtained further interesting series related to these constants. An advantage of our approach is that one can even derive such series representations to be valid for specific regions in the complex plane. As a second goal, we study certain exponential type series rapidly converging to the special values of L-functions and the Riemann zeta function. Precisely, our first result is as follows.

Theorem 1. *Let $\mathfrak{F}_1$ be a region in the complex plane defined by the conditions $|\sin 2z|\leqslant 1$ and $|\Re(z)|\leqslant\frac{\pi}{4}$. Then for any $z\in\mathfrak{F}_1$, we have*

$$\sum_{n=1}^{\infty}\frac{(2\sin 2z)^{2n}}{\binom{2n}{n}n^2(2n+1)^2}=-12+8z^2+16z\cot 2z+\frac{8}{\sin 2z}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(\tan z)^{2n-1}}{(2n-1)^2},$$

$$\sum_{n=1}^{\infty}\frac{(2\sin 2z)^{2n}}{\binom{2n}{n}n(2n+1)^2}=4-4z\cot 2z-\frac{4}{\sin 2z}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(\tan z)^{2n-1}}{(2n-1)^2},$$

$$\sum_{n=1}^{\infty}\frac{\Gamma(\frac{n}{2})^2(2\sin 2z)^n}{n!(n+1)^2}=-12+4\pi z+8z^2+16z\cot 2z$$
$$-\frac{2\pi}{\sin 2z}\big(2-\log 2-2\cos 2z+\log(1+\cos 2z)\big)+\frac{8}{\sin 2z}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(\tan z)^{2n-1}}{(2n-1)^2},$$

where Γ is the Gamma function and the principal branch of the logarithm is used.

Let $\mathfrak{F}_2$ be a region in the complex plane defined by the conditions $|\sin z|\leqslant 1$ and $|\Re(z)|\leqslant\frac{\pi}{2}$. Then for any $z\in\mathfrak{F}_2$, we have

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{(\sin z)^{2n+1}}{2^{2n+2}(2n+1)(n+1)^2}=z-\frac{1}{\sin z}\big(2-\log 2-2\cos z+\log(1+\cos z)\big),$$

where the principal branch of the logarithm is used.

Let $\mathfrak{F}_3$ be a strip in the complex plane defined by the condition $|\Re(z)|\leqslant\frac{\pi}{4}$. Then for any $z\in\mathfrak{F}_3$, we have

$$\sum_{n=1}^{\infty}(-1)^{n-1}\binom{2n}{n}\frac{(\tan z)^{2n}}{2^{2n+1}n(2n-1)}=-1+\log 2+\sec z-\log(1+\sec z),$$

where the principal branch of the logarithm is used.

Let us remark that putting $z=x+iy$, the region $\mathfrak{F}_1$ in the above theorem can be described alternatively as the set of all points (x,y) in the plane satisfying the conditions $|x|\leqslant\frac{\pi}{4}$ and $e^{4y}+e^{-4y}-2\cos 4x\leqslant 4$. In particular, if $|x|=\frac{\pi}{4}$, then it follows easily that $(-\frac{\pi}{4},0)$ and $(\frac{\pi}{4},0)$ are the only possible points in this region. A similar description for $\mathfrak{F}_2$ can be given as well. Next let

$$B_m(x)=\sum_{j=0}^{m}\binom{m}{j}B_jx^{m-j}$$

be the mth Bernoulli polynomial, where B_j denotes the jth Bernoulli number. For any real number a, $B_m(a)$ can be obtained from the Taylor series expansion

$$\frac{xe^{ax}}{e^x-1}=\sum_{m=0}^{\infty}\frac{B_m(a)}{m!}x^m$$

about zero, where the radius of convergence is 2π. Such expansions are often useful for studying generating functions of special values of L-functions. The classical formula

$$\frac{x}{e^x-1}=1+\sum_{m=1}^{\infty}(-1)^{m-1}\zeta(1-m)\frac{x^m}{(m-1)!}$$

furnishes a striking example of this. Inspired by certain mock theta function identities arising from Ramanujan's lost notebook (see [7]), Andrews, Urroz and Ono [8] showed that this phenomenon is indeed quite general and they have found