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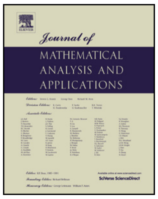

Decay estimates of the coupled chemotaxis–fluid equations in R^3

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ABSTRACT

In this paper, we are concerned with a Chemotaxis–Navier–Stokes model, arising from biology, which is a coupled system of the chemotaxis equations and the viscous incompressible fluid equations with transport and external force. The optimal convergence rates of classical solutions to the Chemotaxis–Navier–Stokes system for small initial perturbation around constant states are obtained by pure energy method under the assumption the initial data belong to $\dot{H}^{-s} \cap H^N$, $N \geqslant 3$ $(0 \leqslant s < 3/2)$. The $\dot{H}^{-s}$ $(0 \leqslant s < 3/2)$ negative Sobolev norms are shown to be preserved along time evolution. Compared to the result in [5], we obtain the optimal decay rates of the higher-order spatial derivatives of the solutions.

1. Introduction

Chemotaxis is a biological process in which cells (e.g., bacteria) move towards a chemically more favorable environment. For example, bacteria often swim towards higher concentration of oxygen to survive. Generally, the motion of the fluid is determined by the well-known incompressible Navier–Stokes equations or Stokes equations. Thus, this kind of cell–fluid interaction becomes more complicated since it not only consists of chemotaxis and diffusion, but also includes transport and viscous fluid dynamics. In particular, it is interesting and important in biology to study some phenomenon of sedimentation on the basis of the coupled cell–fluid model. In [11], the authors observed large-scale convection patterns in a water drop sitting on a glass surface containing oxygen-sensitive bacteria, oxygen diffusing into the drop through the fluid-air interface and they proposed this model:

$$\begin{cases} \partial_t n + u \cdot \nabla n = \delta \Delta n - \nabla \cdot \big(\chi(c) n \nabla c\big), \\ \partial_t c + u \cdot \nabla c = \mu \Delta c - k(c) n, \\ \partial_t u + u \cdot \nabla u + \nabla P = \nu \Delta u - n \nabla \phi, \\ \nabla \cdot u = 0, \quad t > 0,\ x \in R^3. \end{cases} \tag{1}$$

Here, the unknowns are $n = n(t,x)$, $c = c(t,x)$, $u = u(t,x)$, $P = P(t,x)$ denoting the cell density, chemical concentration, velocity field and pressure of the fluid, respectively. Ω is a domain where the cells and the fluid move and interact. Positive constants δ, μ and ν are the corresponding diffusion coefficients for the cells, chemical and fluid. $\chi(c)$ is the chemotactic

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sensitivity and $k(c)$ is the consumption rate of the chemical by the cells. $\phi(x)$ is a given potential function accounting the effects of external forces such as gravity. The system (1) is supplied with initial conditions

$$(n, c, u)|_{t=0} = \big(n_0(x), c_0(x), u_0(x)\big), \quad x \in \Omega,$$

and some proper boundary conditions. The experimental set-up corresponds to mixed-type boundary conditions [11]. However, we work in full space R^3.

In this paper, we consider the decay rates of (1) by using an energy method under the assumption that the initial datum (n, c, u) is a small smooth perturbation of the constant state $(\hat{n}, 0, 0)$ with $\hat{n} > 0$. In [5] Duan et al. not only obtain the global solution for small smooth perturbation but also get the optimal rates of (1) by L^p estimates method under suitable assumption. In [5], under the assumption $(n_0 - \hat{n}, c_0) \in L^1(R^3)$ $u_0 \in L^q(R^3)$ and $\phi \in L^\infty(R^+; L^{\frac{2q}{2-q}}(R^3))$, $q \in (1, \frac{6}{5})$ they get the decay rate

$$\begin{cases} \|n - \hat{n}\|_{L^p} \leqslant C\|n_0 - \hat{n}\|_{L^1 \cap L^p}(1+t)^{-\frac{3}{2}(1-\frac{1}{p})}, \quad 1 \leqslant p < \infty, \\ \|c\|_{L^p} \leqslant C\|c_0\|_{L^1 \cap L^p}(1+t)^{-\frac{3}{2}(1-\frac{1}{p})}, \quad 1 \leqslant p < \infty, \\ \|u\|_{L^2} \leqslant C\big(\|u_0\|_{L^1 \cap H^3} + \|(n_0 - \hat{n}, c_0)\|_{L^1 \cap H^3} + \|n_0 - \hat{n}\|_{L^1 \cap L^2}\|c_0\|_{L^1 \cap L^2}\big)(1+t)^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{2})}. \end{cases} \tag{2}$$

For $2 \leqslant p < \infty$, the decay rates of cell density n and chemical concentration c were obtained by L^p energy method in [7]. The decay estimates of cell density n and chemical concentration c for $1 \leqslant p \leqslant 2$ are obtained because they can directly get $(n, c)(t) \in L^1$ under the initial data $(n, c)(0) \in L^1$. By interpolation method, they get the decay estimates of cell density n and chemical concentration c for all $1 \leqslant p < \infty$.

By using the pure energy method similar to [12] and supplying with initial data in different $H^3 \cap \dot{H}^{-s}$ $(0 \leqslant s < \frac{3}{2})$ spaces, we find the (1) enjoy the same decay rates for L^p norm. But we extend the velocity with the same decay rate for all $2 \leqslant p < \infty$ and the nonlinear convective term $u \cdot \nabla u$ in (1) is considered too. The decay rate of higher derivative is obtained too. This method contain two parts: ⟨1⟩ closing the energy estimates at each k-th level (referring to the order of the spatial derivatives of the solution), which will be obtained in Section 4; ⟨2⟩ deriving a novel negative Sobolev estimates for nonlinear system which requires $s < 3/2$, which will be obtained in Section 5.

Next, let us mention some work concerned to this work. For the system (1) and related systems there is a local existence result in [9]. In [5], the authors proved global existence for (1) with the simpler Stokes equations in R^2 and small perturbation Cauchy solution in R^3. In [6], existence issues and asymptotic behaviour are investigated in R^2 or R^3. In [8], the authors obtain the global existence of weak solutions for the chemotaxis-Stokes system with nonlinear diffusion for the cell density.

The Keller–Segel system is the best-studied model for chemotaxis. In the Keller–Segel system, the chemical is produced and not consumed as in our case. For the elliptic–parabolic Keller–Segel model, in [2], the authors summarises the results, i.e. there is a critical mass M, below M they obtained the global existence and above M they got finite-time blow-up. For the parabolic–parabolic Keller–Segel model recent progress has been achieved in [3]. For more references on the general Keller–Segel system, the interested reader can refer to recent work [1,3]. Kinetic models for chemotaxis can be found in [4].

Notation. In this paper, ∇^k with an integer $k \geqslant 0$ stands for the usual any spatial derivatives of order k. $\|f\|_{L^p}$ denotes the usual norm in L^p spaces. We also use $\langle , \rangle$ denote the inner product in L^2 spaces. In other word, $\langle f, g\rangle = \int_{R^3} f \times g\,dx$. C denotes a constant independent of time t, and C_0 denotes a constant only dependent on initial data. $(\bar{n}, 0, 0)$ is the steady state and the $\rho(t) := n(t, x) - \bar{n}$ denotes the density perturbation around the steady state.

$$\Lambda^s f(x) = \int\limits_{\mathbb{R}^3} |\xi|^s \hat{f}(\xi) e^{2\pi i x \cdot \xi}\, d\xi, \tag{3}$$

where $\hat{f}$ is the Fourier transform of f. We define the homogeneous Sobolev space $\dot{H}^s$ of all f for which $\|f\|_{\dot{H}^s}$ is finite, where

$$\|f\|_{\dot{H}^s} := \big\|\Lambda^s f\big\|_{L^2} = \big\||\xi|^s \hat{f}\big\|_{L^2}. \tag{4}$$

Throughout this paper, we also assume the following conditions:

$$\begin{cases} \text{(i)}, & \delta > 0,\ \mu > 0,\ \nu > 0, \\ \text{(ii)}, & n_0(x) \geqslant 0,\ c_0(x) \geqslant 0,\ \nabla \cdot u_0(x) = 0 \text{ for all } x \in R^3, \\ \text{(iii)}, & \chi(\cdot),\ k(\cdot) \text{ and } \phi(\cdot,\cdot) \text{ are smooth with } \chi(0) = k(0) = 0, \text{ and } k'(c) \geqslant 0 \text{ for all } x \in R, \\ \text{(iv)}, & \sup_{t \geqslant 0}\big\|\phi(t, x)\big\|_{L^3} < \infty. \end{cases} \tag{A}$$

Our main results are stated in the following theorem.