J. Math. Anal. Appl. 410 (2014) 55–69

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

The determinants of fourth order dissipative operators with transmission conditions

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ARTICLE INFO

Article history:
Received 23 January 2013
Available online 9 August 2013
Submitted by J. Shi

Keywords:
Dissipative operators
Transmission conditions
Characteristic determinant
Completeness

ABSTRACT

In this paper, discontinuous non-self-adjoint differential operators in Weyl's limit circle are studied. We give the determinant of perturbation connected with the dissipative operator L generated by fourth order differential expression in $L^2(I)$, where discontinuity of operator is dealt with transmission conditions. We obtain the Green's function, then, using characteristic determinant, we prove the completeness of the system of eigenfunctions and associated functions of this dissipative operator.

1. Introduction

Non-self-adjoint spectral problems have more and more applications. For example, interesting non-classical wavelets can be obtained from eigenfunctions and associated functions for non-self-adjoint spectral problems. Thus, such problems are receiving more and more attention, especially the discreteness of the spectrum and the completeness of eigenfunctions.

The non-self-adjointness of spectral problems can be caused by one or more of the following factors: the nonlinear dependence of the problems on the spectral parameter, the non-symmetry of the differential expressions used, and the non-self-adjointness of the boundary conditions involved. Next, we recall some results.

There are several sufficient conditions in which Weyl's limit-circle case holds for a second order differential expression [2, 4–7,14,17]. And the spectral analysis of the second order operator with general boundary conditions have been investigated in detail in [9,10,13,19]. The second order operator with regular boundary conditions and transmission conditions have been investigated in [1,11,16,15,18].

The determinant of perturbation connected with the dissipative operator L generated in $L^2(I)$ by the Sturm–Liouville differential expression in Weyl's limit circle case has been studied by Bairamov and Ugurlu in [3], they using the Livšic theorem, investigated the problem of completeness of the system of eigenfunctions and associated functions of L.

In this paper, a class of discontinuous non-self-adjoint differential operators in Weyl's limit circle are studied. We generalize the results of [3] in fourth order case. We consider the differential expression

$$l(y) = -y^{(4)} + q(x)y, \quad \text{on } I = [a, c) \cup (c, b), \tag{1.1}$$

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0022-247X/$ – see front matter
http://dx.doi.org/10.1016/j.jmaa.2013.08.004

where $I_1 := [a, c)$, $I_2 := (c, b)$ and $I = I_1 \cup I_2$, $q(x)$ is a real-valued function on I and $q(x) \in L^1_{loc}(I)$. The points a and c are regular and b is singular for the differential expression $l(y)$. $q(c\pm) := \lim_{x\to c\pm} q(x)$ one-sided limits exist and are finite. We always assume that Weyl's limit-circle case holds for the differential expression $l(y)$ on I.

We first give the non-self-adjoint boundary conditions and transmission conditions; then, we give the determinant of perturbation connected with the dissipative operator L generated by fourth order differential expression in $L^2(I)$, where discontinuity of operator is dealt with transmission conditions. We obtain the Green's function, then, using characteristic determinant, we prove the completeness of the system of eigenfunctions and associated functions of this dissipative operator.

This paper is organized as follows. In Section 2, we introduce our notation and recall some basic results. The dissipation of the fourth order operator is proved in Section 3. In Section 4, we review the characteristic function and the characteristic determinant, and we get the Green's function. The completeness of eigenfunctions and associated functions is studied in Section 5.

2. Notation and preliminaries

Let

$$\Omega = \{ y \in L^2(I):\ y, y', y'', y''' \in AC_{loc}(I),\ l(y) \in L^2(I) \}.$$

For all $y, z \in \Omega$, we set

$$[y, z]_x = \left(y\bar{z}''' - y'\bar{z}'' + y''\bar{z}' - y'''\bar{z}\right)(x) = -R_{\bar{z}}(x)Q(x)C_y(x), \quad x \in I = [a, c) \cup (c, b),$$

where the bar over a function denotes complex conjugate, and

$$Q(x) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \qquad R_z(x) = \left(z(x), z'(x), z''(x), z'''(x)\right), \qquad C_{\bar{z}}(x) = R_z^*(x),$$

and $R_z^*(x)$ is complex conjugate transpose of $R_z(x)$.

We consider the differential equation

$$-y^{(4)} + q(x)y = \lambda y, \quad x \in I = [a, c) \cup (c, b), \tag{2.1}$$

where λ is a complex parameter.

Let $D(L)$ denote the set of all functions $y \in \Omega$ satisfying the transmission and boundary conditions

$$l_1 y = y(c+) - \alpha_1 y(c-) - \alpha_2 y'''(c-) = 0, \tag{2.2}$$

$$l_2 y = y'(c+) - \beta_1 y'(c-) - \beta_2 y''(c-) = 0, \tag{2.3}$$

$$l_3 y = y''(c+) - \beta_3 y'(c-) - \beta_4 y''(c-) = 0, \tag{2.4}$$

$$l_4 y = y'''(c+) - \alpha_3 y(c-) - \alpha_4 y'''(c-) = 0, \tag{2.5}$$

$$l_5 y = \gamma_1 y(a) - \gamma_2 y'''(a) = 0, \tag{2.6}$$

$$l_6 y = \gamma_3 y'(a) - \gamma_4 y''(a) = 0, \tag{2.7}$$

$$l_7 y = [y, z_{12}]_b - h[y, z_{42}]_b = 0, \qquad \Im h > 0, \tag{2.8}$$

$$l_8 y = [y, z_{22}]_b - k[y, z_{32}]_b = 0, \qquad \Im k < 0, \tag{2.9}$$

where h and k are some complex numbers with $\Im h > 0$, $\Im k < 0$, α_i, β_i, γ_i $(i = 1, 2, 3, 4)$ are real numbers with $\gamma_1\gamma_3 \neq 0$ and

$$\begin{vmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{vmatrix} = \begin{vmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{vmatrix} = \rho > 0,$$

and $z_{12}(x)$, $z_{22}(x)$, $z_{32}(x)$, $z_{42}(x)$ are the solutions of equation $l(y) = 0$ on interval $I_2 = (c, b)$, it will be given later.

For convenience, we let

$$C = \begin{pmatrix} \alpha_1 & 0 & 0 & \alpha_2 \\ 0 & \beta_1 & \beta_2 & 0 \\ 0 & \beta_3 & \beta_4 & 0 \\ \alpha_3 & 0 & 0 & \alpha_4 \end{pmatrix},$$