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The mean transform of bounded linear operators

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ABSTRACT

In this paper we introduce the mean transform of bounded linear operators acting on a complex Hilbert space and then explore how the mean transform of weighted shifts behaves, in comparison with the Aluthge transform.

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1. Introduction

Let $\mathcal{H}$ be an infinite dimensional complex Hilbert space and $\mathcal{B}(\mathcal{H})$ be the algebra of bounded linear operators acting on $\mathcal{H}$. For $T \in \mathcal{B}(\mathcal{H})$, let $T = U|T|$ be the polar decomposition of T. The *Aluthge transform* $\widetilde{T}$ of T is defined by $\widetilde{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$. This transform was first studied in [1] and has received much attention in recent years, in particular, in relation to the invariant subspace problem. The *Duggal transform* $\widetilde{T}^D$ of T is defined by $\widetilde{T}^D = |T|U$, which is first referred to in [12]. Clearly, the spectrum of $\widetilde{T}$ (resp. $\widetilde{T}^D$) equals that of T. For $\alpha \equiv \{\alpha_k\}_{k=0}^{\infty}$ a bounded sequence of positive real numbers (called *weights*), let $W_\alpha \equiv \text{shift}(\alpha_0, \alpha_1, \ldots) : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$ be the associated (*unilateral*) *weighted shift*, defined by $W_\alpha e_k := \alpha_k e_{k+1}$ (all $k \geqslant 0$), where $\{e_k\}_{k=0}^{\infty}$ is the canonical orthonormal basis in $\ell^2(\mathbb{Z}_+)$. If $\widetilde{W}_\alpha$ is the Aluthge transform of W_α, then we can see that $\widetilde{W}_\alpha = \text{shift}(\sqrt{\alpha_0\alpha_1}, \sqrt{\alpha_1\alpha_2}, \ldots)$, where we note that each term of weights of $\widetilde{W}_\alpha$ consists of the geometric mean of two consecutive terms of W_α. In this paper we introduce a new transform: if $T = U|T|$ is the polar decomposition of T, then we define

$$\widehat{T} := \frac{1}{2}\big(U|T| + |T|U\big) \equiv \frac{1}{2}\big(T + \widetilde{T}^D\big),$$

which will be called the *mean transform* of T and then examine various questions on the mean transform. In particular we will focus on the mean transform of weighted shifts. If $\widehat{W}_\alpha$ is the mean transform of the weighted shift $W_\alpha = \text{shift}(\alpha_0, \alpha_1, \ldots)$, then we can see that $\widehat{W}_\alpha = \text{shift}(\frac{\alpha_0+\alpha_1}{2}, \frac{\alpha_1+\alpha_2}{2}, \ldots)$ (see Proposition 2.2 below). In comparison with

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the Aluthge transform of weighted shifts, the weights of the mean transform of weighted shifts consist of the arithmetic means of two consecutive weights of W_α. This suggests there would be a significant difference or resemblance between the Aluthge transform and the mean transform. First of all, we list problems in which we are interested:

Problem 1.1. Does the spectrum of $\widehat{T}$ equal that of T?

Problem 1.2. Given the mean transform map $T \to \widehat{T}$, (i) is it $(\|\cdot\|, \|\cdot\|)$-continuous on $\mathcal{B}(\mathcal{H})$?; (ii) is it $(\|\cdot\|, SOT)$-continuous on $\mathcal{B}(\mathcal{H})$?

Problem 1.3. For $k \geqslant 1$, if W_α is k-hyponormal, does it follow that the mean transform $\widehat{W_\alpha}$ is also k-hyponormal?

Problem 1.4. If W_α is subnormal with Berger measure μ, does it follow that $\widehat{W_\alpha}$ is subnormal? If it does, what is the Berger measure of $\widehat{W_\alpha}$?

In Section 2 we provide basic properties of the mean transform $\widehat{T}$. In Section 3 we consider the k-hyponormality and the subnormality for the mean transform of the weighted shifts and moreover the continuity properties of the mean transform.

2. Basic properties of the mean transform $\widehat{T}$

If $\widehat{T}$ is the mean transform of T, then we can easily check that $\|\widehat{T}\| \leqslant \|T\|$ in general. How about the spectrum of $\widehat{T}$? It is well known that the spectrum of the Aluthge transform $\widetilde{T}$ (resp. the Duggal transform) equals that of T. We may ask what happens for the spectrum of the mean transform $\widehat{T}$ of T. We first give an answer for Problem 1.1. For this, we let $P \in \mathcal{B}(\mathcal{H})$ be a positive operator and consider an operator matrix $T := \left(\begin{smallmatrix} 0 & P \\ 0 & 0 \end{smallmatrix}\right) \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$. Then $\sigma(T) = \{0\}$. A direct calculation shows that $T = U|T|$, with $U := \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$ and $|T| := \left(\begin{smallmatrix} 0 & 0 \\ 0 & P \end{smallmatrix}\right)$. We thus have that $\widehat{T} = \frac{1}{2}\left(\begin{smallmatrix} 0 & P \\ P & 0 \end{smallmatrix}\right)$. Observe that

$$\widehat{T}^2 = \frac{1}{4}\begin{pmatrix} P^2 & 0 \\ 0 & P^2 \end{pmatrix}, \quad \text{and hence} \quad \sigma\bigl(\widehat{T}^2\bigr) = \left\{\frac{\sigma(P^2)}{4}\right\},$$

which implies $\sigma(\widehat{T}) = \{\pm\frac{\sigma(P)}{2}\}$. Thus we obtain:

Example 2.1. Let $T := \left(\begin{smallmatrix} 0 & P \\ 0 & 0 \end{smallmatrix}\right) \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$, where $P \in \mathcal{B}(\mathcal{H})$ is a positive operator. Then we have

(i) $\sigma(T) = \{0\}$;
(ii) $\sigma(\widehat{T}) = \{\pm\frac{\sigma(P)}{2}\}$.

Hence, in particular, $\sigma(T) \neq \sigma(\widehat{T})$ if $P \neq 0$, while $\|\widehat{T}\| \leqslant \|T\|$.

Since the Duggal transform $\widetilde{T}^D$ shares many spectral properties with T (besides $\sigma(T) = \sigma(\widetilde{T}^D)$) and $\widehat{T} = \frac{1}{2}(T + \widetilde{T}^D)$, one might be tempted to guess that $\sigma(T) \subseteq \sigma(\widehat{T})$. But Example 2.1 illustrates that this is not such a case: consider the case $P = I$. On the other hand, we note that if we define $d(T)$ for the deviation from the normaloid-ness (normaloid means that norm equals spectral radius) by

$$d(T) := \|T\| - r(T) \quad \bigl(\text{where } r(T) \text{ denotes the spectral radius of } T\bigr),$$

then T in Example 2.1 has $d(\widehat{T}) = 0$, i.e., $\widehat{T}$ is normaloid, even though $d(T) = \|P\|$. Thus it may happen that $\widehat{T}$ becomes a nice operator (i.e., normaloid) by filling out something (i.e., $r(\widehat{T}) = \frac{r(P)}{2}$, but $r(T) = 0$), but by contrast, the Aluthge transform $\widetilde{T}$ becomes a nice operator by collapsing something (i.e., $\widetilde{T} = 0$, but $T \neq 0$).

The iterated mean transforms (or mean iterates) of an operator T are the operators $\widehat{T}^{(n)}$ $(n \geqslant 0)$, defined by setting $\widehat{T}^{(0)} = T$ and letting $\widehat{T}^{(n+1)}$ be the mean transform of $\widehat{T}^{(n)}$.

We then have:

Proposition 2.2. *For a weighted shift W_α, the mean iterates $\widehat{W}_\alpha^{(n)}$ are also weighted shifts with weight sequences*

$$\alpha^{(n)} \equiv \bigl\{\alpha_i^{(n)}\bigr\}_{i=0}^{\infty} := \left\{\frac{\sum_{j=0}^{n}\binom{n}{j}\alpha_{i+j}}{2^n}\right\}_{i=0}^{\infty}, \tag{2.1}$$

where $\binom{n}{j} = \frac{n!}{j!(n-j)!}$.

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