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Lower previsions induced by filter maps

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ABSTRACT

We investigate under which conditions a transformation of an imprecise probability model of a certain type (coherent lower previsions, n-monotone capacities, minitive measures) produces a model of the same type. We give a number of necessary and sufficient conditions, and study in detail a particular class of such transformations, called filter maps. These maps include as particular models multi-valued mappings as well as other models of interest within imprecise probability theory, and can be linked to filters of sets and $\{0, 1\}$-valued lower probabilities.

1. Introduction

In many practical problems, it is not uncommon to encounter situations with vague or imprecise knowledge about the probabilistic information of the variables involved; this could be due to deficiencies in the observational process or conflicts between the opinions of several experts, amongst other things. In such cases, it may be advisable to consider imprecise probability models as a more robust alternative to the classical models based on probability measures. These models include as particular cases sets of probability measures [14], coherent lower previsions [19], n-monotone capacities [1], and necessity measures [11].

All these models are mathematically related to each other: sets of probability measures and coherent lower previsions are equivalent, and they include n-monotone capacities as special cases. Necessity measures are in particular n-monotone for any natural number n.

Interestingly, a transformation of such models—under a permutation of the possibility space, or more generally, under another map that connects two possibility spaces—need not preserve their character. To give an example, it is not uncommon that a transformation of a necessity measure goes beyond the framework of necessity measures, and produces only a coherent lower prevision (probability). In this paper, we investigate under which conditions a transformation of the possibility space, or of the associated space of real-valued functions defined on it, preserves different consistency notions: avoiding sure loss, coherence, n-monotonicity or being minimum preserving.

We show that for the first two we must consider so-called *coherence preserving* transformations, which are closely related to (but at the same time more general than) coherent lower previsions. We investigate their properties in Sections 3 and 4. We show in Section 7 that as particular cases of coherence preserving mappings, we have the notion of probability induced by a random variable, but also the Markov operators considered in [17]. For the last two conditions, we show in Sections 5 and 6 that a conditional model preserves n-monotonicity in addition to coherence if and only if it is a $\wedge$-homomorphism,

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i.e. minimum preserving. We characterise minimum preserving coherent lower previsions, and show that they are in a one-to-one correspondence with filters of subsets of the possibility space.

This leads us to the second part of the paper, where we study in detail a particular type of mappings, called *filter maps*, which we introduce in Section 8. They are maps that assign to any element of an initial space a filter of subsets of a final space. But they can also be interpreted as minimum preserving transformations. We show that they include as particular cases the lower previsions induced by multi-valued mappings as well as lower oscillation models, and we study the properties of the lower prevision that results from combining a filter map with a lower prevision on the initial space.

2. Preliminaries

2.1. Coherent lower previsions

We begin with a brief discussion of coherent lower previsions. We refer to [19] for more details and background, and for the proofs of all results mentioned in this section.

Consider a possibility space $\mathscr{X}$. A *gamble* f on $\mathscr{X}$ is any bounded real-valued map on $\mathscr{X}$. This is for instance the case for the indicator I_A of a subset A of $\mathscr{X}$, where $I_A(x)$ takes the value 1 when $x \in A$ and 0 when $x \notin A$.

The set of all gambles on $\mathscr{X}$ is denoted by $\mathscr{G}(\mathscr{X})$. This is a linear space: it is closed under point-wise addition and multiplication by real numbers. It is moreover a *lattice*, that is, closed under point-wise maxima $\vee$ and minima $\wedge$. We will be interested in some particular transformations between sets of gambles.

Definition 1. Given two spaces $\mathscr{X}$ and $\mathscr{Y}$, a map $r : \mathscr{G}(\mathscr{X}) \to \mathscr{G}(\mathscr{Y})$ is called a $\wedge$-*homomorphism* when $r(f_1 \wedge f_2) = r(f_1) \wedge r(f_2)$ for all $f_1, f_2 \in \mathscr{G}(\mathscr{X})$.

A real-valued map defined on a set of gambles is called a lower prevision. It can be given a behavioural interpretation: the lower prevision of a gamble f is the supremum price μ such that the transaction $f - \mu$ is desirable for a given subject. This interpretation lies at the basis of the following definitions:

Definition 2. A *lower prevision* on a set of gambles $\mathscr{K} \subseteq \mathscr{G}(\mathscr{X})$ is a functional $\underline{P} : \mathscr{K} \to \mathbb{R}$. It is said to *avoid sure loss* when for every $n \in \mathbb{N}_0$ and $f_1, \dots, f_n \in \mathscr{K}$:

$$\sum_{i=1}^{n} \underline{P}(f_i) \leqslant \sup\left[\sum_{i=1}^{n} f_i\right],$$

where $\mathbb{N}_0$ denotes the set of natural numbers (with zero). It is called *coherent* if and only if for all numbers $n, m \in \mathbb{N}_0$ and $f_0, f_1, \dots, f_n \in \mathscr{K}$:

$$\sum_{i=1}^{n} \underline{P}(f_i) - m\underline{P}(f_0) \leqslant \sup\left[\sum_{i=1}^{n} f_i - mf_0\right].$$

One example of coherent lower previsions are the so-called *vacuous* ones. For any non-empty subset A of $\mathscr{X}$, the vacuous lower prevision relative to A is defined on $\mathscr{G}(\mathscr{X})$ as

$$\underline{P}_A(f) := \inf_{x \in A} f(x) \quad \text{for all } f \in \mathscr{G}(\mathscr{X}).$$

A lower prevision $\underline{P}$ on the set $\mathscr{G}(\mathscr{X})$ of all gambles turns out to be coherent if and only if:

C1. $\underline{P}(f) \geqslant \inf f$ for all gambles f on $\mathscr{X}$;
C2. $\underline{P}(\lambda f) = \lambda \underline{P}(f)$ for all gambles f on $\mathscr{X}$ and all real $\lambda \geqslant 0$;
C3. $\underline{P}(f + g) \geqslant \underline{P}(f) + \underline{P}(g)$ for all gambles f and g on $\mathscr{X}$.

On the other hand, a coherent lower prevision defined on indicators of events only is called a *coherent lower probability*. To simplify the notation, we will sometimes use the same symbol A to denote a set A and its indicator I_A, so we will write $\underline{P}(A)$ instead of $\underline{P}(I_A)$. Of particular interest are the $\{0, 1\}$-valued coherent lower probabilities, which are related to filters of events:

Definition 3. Let $\mathscr{P}(\mathscr{X})$ denote the power set of a space $\mathscr{X}$. A subset $\mathscr{F}$ of $\mathscr{P}(\mathscr{X})$ is called a (proper) *filter* when it satisfies the following properties:

F1. $\emptyset \notin \mathscr{F}$;
F2. if $A, B \in \mathscr{F}$ then $A \cap B \in \mathscr{F}$;
F3. if $A \in \mathscr{F}$ and $A \subseteq B$ then $B \in \mathscr{F}$.