J. Math. Anal. Appl. 410 (2014) 124–133

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

A globally convergent method for computing multiple vortices in a supersymmetric field theory

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ARTICLE INFO

Article history:
Received 17 April 2013
Available online 14 August 2013
Submitted by P. Yao

Keywords:
Non-Abelian gauge field theory
Vortices
Bogomol'nyi equations
Maximum principle
Bounded domain truncation
Iterative scheme
Numerical solutions

ABSTRACT

Computation of the solutions to the gauge field equations is known of great importance for the simulation of various particle physics systems. In this work, we establish a globally convergent iterative method for computing the multiple vortex solutions arising in a self-dual system of non-Abelian gauge field equations derived in a supersymmetric theory model. Using this method, we present a few numerical examples which demonstrate the effectiveness of the method and, at the same time, provide a concrete realization of the soliton-like behavior of the vortexlines concentrated around centers of vortices, which is believed to be essential for linear confinement in QCD.

1. Introduction

The problem of effectively approximating the solutions of gauge field equations has a long and rich history and its progress often leads to significantly improved understanding of various subjects concerned. Notably, a numerical computation of multimeron solutions to the Yang–Mills equations was performed by Jacobs and Rebbi [22] by assuming the rotational symmetry about a fixed axis so that multimeron field configurations are described by a single nonlinear partial differential equation over the plane. They used the covariance properties of this equation under the Möbius transformations and simplified the numerical analysis which enabled them to utilize a variational procedure to evaluate the four-meron solutions and determine their properties. Later, by using a constrained variational calculation, Jacobs and Rebbi [23] were able to determine the interaction energy of two-vortex configurations in the Abelian Higgs model [1,24,37] and showed that when the energy is evaluated as a function of the separation between vortices and of a coupling parameter λ measuring the Higgs boson mass, vortices are attractive for $\lambda < 1$, repulsive for $\lambda > 1$, and noninteracting for $\lambda = 1$. This work preluded the classical study of Taubes [33,34] on noninteracting vortices and laid the foundation for the formulation of a conjecture regarding interacting vortex [24]. This work was later carried over by Jacobs, Khare, Kumar, and Paul [21] and Arthur [7] to the case of Abelian Chern–Simons–Higgs model. In [2], Adler and Piran presented several relaxation and iterative methods for the numerical resolution of elliptic partial differential equations, emphasizing on treating nonlinear problems arising in the effective Lagrangian approximations to the dynamics of quantized gauge fields. In particular, numerical examples simulating a linear electrostatic confinement model [25] are illustrated.

It is well known that the computation of the solutions of gauge field equations encounters a few immediate challenges. The first one is the nonlinearity of the problem due to the interaction of scalar and gauge fields through gauge-covariant

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0022-247X/$ – see front matter
http://dx.doi.org/10.1016/j.jmaa.2013.07.083

derivatives. The second one is the ambiguity or degree of freedom of the solution configurations as a result of the presence of gauge symmetry. The third one is that the equations are often considered to be defined over the full space and, as a consequence, there is a lack of compactness. Therefore, any effective computational method for the solution of gauge field equations has to be able to resolve these three issues.

The purpose of our work in the present article is to establish an effective and globally convergent computation method to solve the multiple vortex equations arising in the supersymmetric gauge field theory model of Eto, Fujimori, Nagashima, Nitta, Ohashi, and N. Sakai [14], and mathematically studied recently by Lin and Yang [26]. The solutions of these equations, known as multiple vortices, are relevant in quark confinement [8,9,13–15,18,28–30] in QCD (quantum chromodynamics). In our context, the difficulties with nonlinearity, complicated coupling of scalar and gauge fields, and gauge freedom are overcome by a reduction of the equations of motion into a BPS system [11,27]. For example, within such a reduction, the governing equations are expressed in terms of the amplitudes of the Higgs scalar fields so that the ambiguity associated with gauge symmetry is automatically removed. Moreover, the difficulty with the full plane setting will be seen to be removed by utilizing the asymptotic decay properties of the solutions so that the solutions may be effectively approximated by a sequence of bounded domain truncations, first observed in [36] in a single-scalar equation setting modeling the Abrikosov vortices [1] in the Ginzburg–Landau theory [16]. In our situation here, the problem is more complicated due to the non-Abelian nature of the model in that the equations governing multiple vortices are composed of two coupled nonlinear elliptic equations. Fortunately, employing the analytic techniques developed in [26], we are able to obtain a globally convergent approximation method for computing these multiple vortex solutions.

In addition to the classical areas of applications of vortices, such as superconductivity [1,16,24] and particle physics [19,20,37], it is known that vortices have important applications in many other fundamental areas of physics including condensed matter [12,37], laser optics [10], and cosmology [35,37]. It will be of future interest to carry out our work to those areas.

An outline of the rest of the paper is as follows. In Section 2, we review and describe the system of elliptic equations governing multiple vortex solutions in the supersymmetric model [14]. In Section 3, we present our iterative scheme and prove its local and global convergence. In Section 4, we demonstrate a few concrete numerical examples to illustrate the effectiveness of our method. These examples confirm the soliton-like behavior of the vortexlines concentrated around the centers of the vortices. In particular, we observe the influence of the coupling parameters on the field configurations. Section 5 is a brief summary.

2. The system of governing equations

Use W_μ and w_μ to denote the two gauge fields taking values in the Lie algebras of the unitary gauge groups $U(N)$ ($N \geqslant 2$) and $U(1)$, respectively, and H a complex $N \times N$ matrix representing N Higgs scalar fields in the fundamental representation of the gauge groups. Use

$$\mathcal{D}_\mu H = \partial_\mu H + \mathrm{i} W_\mu H + \mathrm{i} w_\mu H \tag{2.1}$$

to denote the gauge-covariant derivatives,

$$F_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu + \mathrm{i}[W_\mu, W_\nu], \qquad f_{\mu\nu} = \partial_\mu w_\nu - \partial_\nu w_\mu, \tag{2.2}$$

the associated curvature tensors, and $\langle X\rangle = X - \mathrm{Trace}(X) I_N$ to represent the trace-free part of an $N \times N$ matrix X. Then the Lagrangian action density $\mathcal{L}$ of the supersymmetric gauge field model of Eto, Fujimori, Nagashima, Nitta, Ohashi, and N. Sakai [14] takes the form

$$\mathcal{L} = \mathcal{K} - \mathcal{V}, \tag{2.3}$$

$$\mathcal{K} = \mathrm{Trace}\left(-\frac{1}{2g^2} F_{\mu\nu} F^{\mu\nu} + \mathcal{D}H\big[\mathcal{D}^\mu H\big]^\dagger\right) - \frac{1}{4e^2} f_{\mu\nu} f^{\mu\nu}, \tag{2.4}$$

$$\mathcal{V} = \frac{g^2}{4}\mathrm{Trace}\big(\langle HH^\dagger\rangle^2\big) + \frac{e^2}{2}\big(\mathrm{Trace}\big[HH^\dagger - cI_N\big]\big)^2, \tag{2.5}$$

where g, e, c are positive coupling constants and the way of their appearance in the action density makes the theory enjoy a critical BPS reduction so that the second-order equations of motion in the static vortex case are replaced by the following first-order equations:

$$\mathcal{D}_1 H + \mathrm{i}\mathcal{D}_2 H = 0, \tag{2.6}$$

$$F_{12} = \frac{m_g^2}{2c}\langle HH^\dagger\rangle, \tag{2.7}$$

$$f_{12} = \frac{m_e^2}{2c}\mathrm{Trace}\big(HH^\dagger - cI_N\big), \tag{2.8}$$