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Uncertainty principles in finitely generated shift-invariant spaces with additional invariance

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ABSTRACT

We consider finitely generated shift-invariant spaces (SIS) with additional invariance in $L^2(\mathbb{R}^d)$. We prove that if the generators and their translates form a frame, then they must satisfy some stringent restrictions on their behavior at infinity. Part of this work (non-trivially) generalizes recent results obtained in the special case of a principal shift-invariant spaces in $L^2(\mathbb{R})$ whose generator and its translates form a Riesz basis.

1. Introduction

Finitely generated shift-invariant spaces have been widely used in approximation theory, numerical analysis, sampling theory and wavelet theory (see e.g. [1,3,5,10,12,13,16,17] and the references therein). Shift-invariant spaces with additional invariance have been studied in the context of wavelet analysis and sampling theory [4,11,14,18], and have been given a complete algebraic description in [2] for $L^2(\mathbb{R})$ and in [7] for $L^2(\mathbb{R}^d)$. As a tool for showing our main results, we will prove a slightly different but useful characterization.

Definition 1.1. A closed subspace V of $L^2(\mathbb{R}^d)$ is called a *shift-invariant space* if $f(\cdot - t) \in V$ for any $f \in V$ and any $t \in \mathbb{Z}^d$. V is called a *translation-invariant space* if $f(\cdot - t) \in V$ for any $f \in V$ and any $t \in \mathbb{R}^d$.

It is well known that the Paley–Wiener space *PW* is translation-invariant. Moreover, Shannon's sampling theorem easily implies that *PW* is principal, i.e., generated by the single function sinc. It turns out that the fact that sinc is non-integrable is not a coincidence. Actually, for a principal shift-invariant space in $L^2(\mathbb{R})$ which is translation-invariant, any frame generator is non-integrable (see for instance [6]). This observation holds in any dimension. Indeed, the Fourier transform $\widehat{\phi}$ of a frame generator has to satisfy for a.e. $\omega \in \mathbb{R}^d$

$$C^{-1} 1_A(\omega) \leqslant \left|\widehat{\phi}(\omega)\right| \leqslant C 1_A(\omega)$$

for some $C \geqslant 1$ and some finite measure subset A. In particular, $\widehat{\phi}$ is not continuous. Such a condition also prevents $\widehat{\phi}$ from being in the Sobolev space[1] $H^{\frac{1}{2}}(\mathbb{R}^d)$ (see [15]), whereas it belongs to $H^{\frac{1}{2}-\epsilon}(\mathbb{R}^d)$ for every $\epsilon > 0$ when A is a Euclidean ball of positive radius.

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[1] I.e., $\int |\phi(x)|^2 (1 + |x|)\, dx = \infty$.

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Our first result is a straightforward generalization of this fact to shift-invariant spaces generated by several functions.

Theorem 1.2. *Let Λ be a lattice in $\mathbb{R}^d$. If a finitely generated Λ-invariant space of $L^2(\mathbb{R}^d)$ is translation-invariant, then at least one of its frame generators has a non continuous Fourier transform, and in particular is not in L^1. Moreover this generator satisfies $\int |\phi(x)|^2(1+|x|)\,dx = \infty$.*

The slow spatial-decay of the generators of shift-invariant spaces that are also translation-invariant is a disadvantage for the numerical implementation of some analysis and processing algorithms. This is a motivation for considering instead shift-invariant spaces that are only $\frac{1}{n}\mathbb{Z}$-invariant and hoping the generators will have better time-frequency localization. Indeed, it was shown in [6] that for every n, one can construct a principal shift-invariant space with an orthonormal generator which is in $L^1(\mathbb{R})$ although the space is $\frac{1}{n}\mathbb{Z}$-invariant. However, there are still obstructions if we require more regularity on the Fourier transform of the generators. Asking for fractional differentiability yields the following Balian–Low type obstructions (see [8] and the reference therein). Compare [6, Theorem 1.2].

Theorem 1.3. *Let $\Lambda < \Gamma$ be two lattices*[2] *in $\mathbb{R}^d$. Suppose $\phi_i \in L^2(\mathbb{R}^d)$ are such that $\{\phi_i(\cdot+\lambda) \mid \lambda \in \Lambda,\ i = 1,\dots,r\}$ forms a frame for the closed subspace $V^\Lambda(\Phi)$ spanned by these functions. Let ρ ($\leqslant r$) be the minimal number of generators of $V^\Lambda(\Phi)$. Assume that $[\Gamma : \Lambda]$ is not a divisor of ρ, and suppose that $V^\Lambda(\Phi)$ is Γ-invariant. Then there exists $i_0 \in \{1,\dots,r\}$ such that*

$$\int\limits_{\mathbb{R}^d} \left|\phi_{i_0}(x)\right|^2 |x|^{d+\epsilon}\,dx = +\infty$$

for all $\epsilon > 0$. I.e., $\widehat{\phi}_{i_0}$ is not in $H^{\frac{d}{2}+\epsilon}(\mathbb{R}^d)$.

One can also ask for a combination of regularity, namely continuity and of control of the decay at infinity. In this spirit, we obtain the following result (compare [6, Theorem 1.3]).

Theorem 1.4. *Keep the same assumptions as in the last theorem, and suppose moreover that $\widehat{\phi}_i$ is continuous for every $i = 1,\dots,r$. Then there exits $i_0 \in \{1,\dots,r\}$ such that $\omega^{\frac{d}{2}+\epsilon}\widehat{\phi}_{i_0}(\omega) \notin L^\infty(\mathbb{R}^d)$ for any $\epsilon > 0$, i.e.,*

$$\operatorname*{ess\,sup}_{\omega\in\mathbb{R}} \left|\widehat{\phi}_{i_0}(\omega)\right| |\omega|^{\frac{d}{2}+\epsilon} = +\infty.$$

For $d = 1$, the exponent is sharp up to the ϵ in both theorems but this does not seem to be the case for larger d. We actually conjecture that the right exponent should be the same for all dimensions.

Notice that the condition $[\Gamma : \Lambda] \nmid \rho$ in the above theorems is essential. Indeed all regularity constraints trivially disappear when $\rho = k[\Gamma : \Lambda]$ for any integer $k > 0$. To see why, start with some Γ-invariant space generated by exactly k orthogonal generators $\phi_1,\dots,\phi_k$, with – say – smooth and compactly generated Fourier transforms. Then note that as a Λ-invariant space, $V^\Gamma(\phi_1,\dots,\phi_k)$ is generated by the orthogonal generators $\phi_i(\cdot - f)$ for $f \in F$ and $i = 1,\dots,k$, where F is a section of Γ/Λ in Γ.

The previous results state that under additional invariance there always exists at least one (frame) generator whose Fourier transform has poor regularity. One may wonder if at least some generators can be chosen with good properties. We do not know what the optimal proportion of good generators should be. The following proposition gives a lower bound on the number of good generators. Observe that this bound gets worse with the dimension.

Proposition 1.5. *For every $d \geqslant 1$, and every $k \in \mathbb{N}$ there exists an SIS $V(\Phi)$ in $L^2(\mathbb{R}^d)$ generated by an orthonormal basis Φ consisting of $r = (2k)^d$ functions, k^d of which have smooth and compactly supported Fourier transforms, and such that $V(\Phi)$ is translation-invariant. Moreover all the generators can be chosen so that their Fourier transforms are in $H^{\frac{1}{2}-\epsilon}(\mathbb{R}^d)$ for all $\epsilon > 0$.*

We also have the following result,

Proposition 1.6. *For $d \geqslant 1$, let $\Gamma (> \mathbb{Z}^d)$ be a lattice of $\mathbb{R}^d$ and $r \geqslant 1$. Then there exists an SIS $V(\Phi)$ in $L^2(\mathbb{R}^d)$ generated by r orthonormal generators ϕ_i's, all of which are in L^1 (hence they have continuous Fourier transforms) and satisfy $\omega^{\frac{1}{2}}\widehat{\phi}_i(\omega) \in L^\infty(\mathbb{R}^d)$. Moreover, $V(\Phi)$ is Γ-invariant.*

To summarize, while Proposition 1.5 states that it is possible to construct translation-invariant SIS with a portion of the generators having smooth and compactly supported Fourier transforms, Proposition 1.6 shows that we can construct Γ-invariant SIS with all its generators having certain pointwise decay in Fourier domain.

[2] The notation $\Lambda < \Gamma$ is standard to denote subgroups.